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Estimates for the discrete fundamental solution of the discrete Laplace operator on a rectangular lattice

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This paper presents numerical analysis of the discrete fundamental solution of the discrete Laplace operator on a rectangular lattice. Additionally, to provide estimates in interior and exterior domains, two different regularisations of the discrete fundamental solution are considered. Estimates for the absolute difference and l^p-estimates are constructed for both regularisations. Thus, this work extends the classical results in the discrete potential theory to the case of a rectangular lattice and serves as a basis for future convergence analysis of the method of discrete potentials on rectangular lattices.

KEYWORDS

discrete fourier transform, discrete fundamental solution, discrete potential theory, estimates, laplace operator, rectangular lattice

MSC CLASSIFICATION

39A12; 35A08; 65E05

1 | INTRODUCTION

The tools of classical potential theory are used in various applications to address problems of mathematical physics. As usual in the use of analytical approaches, their applications to real-world problems involve numerical schemes to solve boundary integral equations arising from potential-theoretic approaches. The use of numerical schemes implies that important model properties, such as for example conservation laws, are only approximated and not satisfied on the discrete level. To overcome this obstacle, constriction of discrete analogues of the continuous potential theory has been an area of active research for several decades.[1,2]

Different approaches to discrete potential theory and its applications has been studied by several authors in recent years.[3–7] Typically, the classical setting of a square lattice have been addressed so far. However, for practical applications of discrete potential theory, a more general type of lattices would be advantageous. As a first step towards generalising the discrete potential theory in the sense of Hommel[1] to a more type of lattices, a rectangular lattice allowing two different stepsizes h_1 and h_2 has been proposed recently.[8–10] Although introducing two stepsizes instead of one stepsize, as in the classical case, at first look makes impression of a mild change to the theory, it is not the case in reality. The problem comes from the fact that the discrete fundamental solution of the discrete Laplace operator, which is the core of the discrete potential theory, on a rectangular lattice cannot be obtained from the discrete fundamental solution on a square lattice.[8] Thus, results from the discrete potential theory on square lattices cannot be directly transferred to the rectangular case and, therefore, needs to be extended.

The goal of this paper is to present results in numerical analysis of the discrete fundamental solution of the discrete Laplace operator on a rectangular lattice. Precisely, error estimates (pointwise difference and l^p-estimates) for two regularisations of the discrete fundamental solutions are constructed for interior and exterior settings. The reason for using two regularisation is motivated by the fact, that the first regularisation suits better for interior problems, while the second one allows consideration of exterior problems. All estimates are constructed explicitly, and plots illustrating the estimates are provided. Thus, this work provides a basis for studying convergence of discrete potentials on rectangular lattice, which is the scope of future work.

2 | FUNDAMENTAL SOLUTION OF THE DISCRETE LAPLACE OPERATOR ON A RECTANGULAR LATTICE

Let us denote by $\mathbb{R}^2_{h_1,h_2} := \left\{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = (m_1h_1, m_2h_2), m_j \in \mathbb{Z}, j = 1,2\right\}$ an unbounded *rectangular* lattice in $\mathbb{R}^2$ with two lattice constants $h_1, h_2 > 0$. We introduce now the shift operators[8,10] $S_{\pm j}\mathbf{x} := \mathbf{x} + h_{|j|}\mathbf{e}_{\pm j}$, $\mathbf{x} \in \mathbb{R}^2$, with $\mathbf{e}_j$, $j = 1,2$ being the standard unit vectors in $\mathbb{R}^2$ and the convention $\mathbf{e}_{-j} = -\mathbf{e}_j$. Next, we introduce the classical finite difference operators $D_j = \frac{1}{h_j}(S_j - I)$ and $D_{-j} = \frac{1}{h_j}(I - S_{-j})$ for $j = 1,2$, where I denotes the identity operator. The discrete Laplace operator Δ_{h_1,h_2} on a rectangular lattice is then introduced as follows:

$$\Delta_{h_1,h_2} := \sum_{j=1}^{2} D_{-j}D_j = \sum_{j=1}^{2} D_jD_{-j}.$$

The discrete fundamental solution is introduced in the following definition:

Definition 1. The function E_{h_1,h_2} is called a *discrete fundamental solution* of the discrete Laplace operator Δ_{h_1,h_2} if it satisfies

$$-\Delta_{h_1,h_2}E_{h_1,h_2}(\mathbf{x}) = \delta_{h_1,h_2}(\mathbf{x}) \tag{1}$$

for all mesh points $\mathbf{x} = (m_1h_1, m_2h_2)$ of $\mathbb{R}^2_{h_1,h_2}$, where $\delta_{h_1,h_2}(\mathbf{x})$ is the discrete Dirac delta function defined as follows

$$\delta_{h_1,h_2}(\mathbf{x}) := \begin{cases} \frac{1}{h_1h_2}, & \text{for } \mathbf{x} = (0,0), \\ 0, & \text{for } \mathbf{x} \neq (0,0). \end{cases}$$

The discrete fundamental solution E_{h_1,h_2} is constructed by help of the discrete Fourier transform on a rectangular lattice,[8,11] and it is given by the following integral representation:

$$E_{h_1,h_2}(m_1h_1, m_2h_2) = \left(\frac{1}{2\pi}\right)^2 \int_{-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \int_{-\frac{\pi}{h_1}}^{\frac{\pi}{h_1}} \frac{e^{-i(m_1h_1y_1+m_2h_2y_2)} - 1}{d^2} dy_1dy_2, \tag{2}$$

which is the *discrete fundamental solution of the discrete Laplace operator on a rectangular lattice*, and $d^2 = \frac{4}{h_1^2}\sin^2\frac{h_1y_1}{2} + \frac{4}{h_2^2}\sin^2\frac{h_2y_2}{2}$ is the corresponding Fourier symbol. A discussion on the relation between the discrete fundamental solution (2) and the classical discrete fundamental solution on a rectangular lattice has been done already,[10] and it has been underlined that the fundamental solution on a rectangular lattice cannot be obtained from the fundamental solution on a square lattice. For the convergence analysis of the discrete fundamental solution $E_{h_1,h_2}(m_1h_1, m_2h_2)$ the following regularisation of the continuous solution will be used[12]:

$$E(x) = \frac{1}{(2\pi)^2}\left(\int_{|\mathbf{y}|<1} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}} - 1}{\mathbf{y}^2}d\mathbf{y} + \int_{|\mathbf{y}|>1} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}}{\mathbf{y}^2}d\mathbf{y}\right) = -\frac{1}{2\pi}(C - \ln 2 - \ln|\mathbf{x}|), \tag{3}$$

where $C = \lim\limits_{n\to\infty}\left(-\ln n + \sum\limits_{k=1}^{n}\frac{1}{k}\right) \approx 0.57721$ is the Euler constant.

First steps in convergence analysis will be performed by working with the following regularised form of the discrete fundamental solution

$$E^{(1)}_{h_1,h_2}(\mathbf{x}) = \frac{1}{(2\pi)^2}\left(\int\limits_{|\mathbf{y}|<1} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}-1}{d^2}d\mathbf{y} + \int\limits_{|\mathbf{y}|>1, \mathbf{y}\in Q_{h_1,h_2}} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}}{d^2}d\mathbf{y}\right), \text{ with } Q_{h_1,h_2} := \left\{ \mathbf{y}\in\mathbb{R}^2 | -\frac{\pi}{h_j} < y_j < \frac{\pi}{h_j}, j=1,2 \right\}, \quad (4)$$

where it has been taken into account that the convergence analysis is of interest here, i.e., $h_1, h_2 \to 0$, and therefore, the interior of unit disk $|y|<1$ needs to lay inside the rectangle Q_{h_1,h_2}, meaning that $h_1 < \pi$ and $h_2 < \pi$. The discrete fundamental solution $E^{(1)}_{h_1,h_2}(\mathbf{x})$ differs from the discrete fundamental solution (2) by the following expression depending on h_1 and h_2:

$$K_1 = \frac{1}{(2\pi)^2} \int\limits_{|\mathbf{y}|>1, \mathbf{y}\in Q_{h_1,h_2}} \frac{1}{d^2} d\mathbf{y}.$$

3 | ESTIMATES FOR THE DISCRETE FUNDAMENTAL SOLUTION OF THE DISCRETE LAPLACE OPERATOR

It is important to remark that formula (2) provides a general form of the discrete fundamental solution of the discrete Laplace operator on a rectangular lattice. However, for analysis of this fundamental solution, similar to the continuous case,[12] it is necessary to work with a regularised version of it. One of possible regularisations is provided by formula (4), which, as it will be shown in this section, is suitable for constructing error estimates in the interior setting, but not for the exterior setting. Therefore, the estimates in the exterior setting will be constructed by working with another regularisation of the discrete fundamental solution (2), which will be introduced in Section 3.2. For both regularisations, estimates for the pointwise difference to the continuous fundamental solution, as well as difference in l^p space, will be presented in this section.

3.1 | Estimates for the discrete fundamental solution $E^{(1)}_{h_1,h_2}$

Let us start with the following theorem:

Theorem 1. *Let $E^{(1)}_{h_1,h_2}$ be the discrete fundamental solution given in (4) of the discrete Laplace operator, and let E be the continuous fundamental solution (3) of the classical Laplace operator. Then for all $\mathbf{x}\neq 0$ and all $h_1, h_2 < \sqrt{2}\pi$ the following estimate holds*

$$|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| \leq C_1 \max\left\{h_1^2, h_2^2\right\} + \frac{C_2}{|\mathbf{x}|}\max\{h_1,h_2\} + \frac{C_3}{|\mathbf{x}|}\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}},$$

where C_1, C_2, and C_3 are arbitrary constants independent on the stepsizes h_1 and h_2.

Proof. The use of definitions of the fundamental solutions and application of the triangle inequality leads to the following:

$$\begin{aligned} |E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| \leq{}& \frac{1}{(2\pi)^2}\left| \int\limits_{|\mathbf{y}|<1} \left(\frac{1}{d^2} - \frac{1}{|\mathbf{y}|^2}\right)(e^{-i\mathbf{x}\cdot\mathbf{y}}-1)d\mathbf{y}\right| \\ &+ \frac{1}{(2\pi)^2}\left| \int\limits_{|\mathbf{y}|>1, \mathbf{y}\in Q_{h_1,h_2}} \left(\frac{1}{d^2} - \frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}\right| + \frac{1}{(2\pi)^2}\left| \int\limits_{|\mathbf{y}|\in\mathbb{R}^2\setminus Q_{h_1,h_2}} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}}{|\mathbf{y}|^2}d\mathbf{y}\right|. \end{aligned} \quad (5)$$

At first, the term $I_1 := \frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|<1}\left(\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}\right)(e^{-i\mathbf{x}\cdot\mathbf{y}}-1)d\mathbf{y}\right|$ will be estimated. Estimation of I_1 requires at first an adaptation of some preliminary results from Stummel[13] to the case of a rectangular lattice. Recalling that for the variables $\xi^j_{h_1,h_2}(y_j)=\frac{1}{h_j}(1-e^{ih_jy_j})$, $j=1,2$ the following equalities are satisfied

$$|\xi^j_{h_1,h_2}(y_j)| = \left|\sqrt{\frac{1}{h_j^2}\left[(1-\cos h_jy_j)^2+\sin^2 h_jy_j\right]}\right| = \left|\frac{1}{h_j}\sqrt{2-2\cos h_jy_j}\right| = \frac{2}{h_j}\left|\sin\frac{h_jy_j}{2}\right|,$$

the following equality for variables $\xi^{-j}_{h_1,h_2}(y_j)=\frac{1}{h_j}(e^{-ih_jy_j}-1)$, $j=1,2$ is obtained then straightforwardly

$$|\xi^{-j}_{h_1,h_2}(y_j)| = \frac{2}{h_j}\left|\sin\frac{h_jy_j}{2}\right|.$$

Finally, by help of the Jordan's inequality $\frac{2}{\pi}x \le \sin x \le x$ for $x\in\left[0,\frac{\pi}{2}\right]$, the following estimates can be obtained

$$\frac{2}{\pi}|y_j| \le |\xi^j_{h_1,h_2}(y_j)| \le |y_j|,\ j=1,2,\ y\in Q_{h_1,h_2}.$$

Now, left inequalities for each j will be considered, and after squaring both sides and adding inequalities for $j=1$ and $j=2$, the following inequality is obtained:

$$\frac{4}{\pi^2}(|y_1|^2+|y_2|^2) \le |\xi^1_{h_1,h_2}(y_1)|^2+|\xi^2_{h_1,h_2}(y_2)|^2,\ \Rightarrow\ \frac{4}{\pi^2}|\mathbf{y}|^2 \le d^2.$$

To estimate the expression $\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}$, the Fourier symbol d^2 will be expanded into Taylor series, and by using the equality $\frac{1}{|\mathbf{y}|^2}\ge\frac{4}{\pi^2 d^2}$, the following estimate is obtained

$$0 \le \frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2} \le \frac{\pi^2}{48}\max\left\{h_1^2,h_2^2\right\}. \tag{6}$$

By using trigonometric identities, as it has been done above, the expression $\left|e^{-i\mathbf{x}\cdot\mathbf{y}}-1\right|$ can be estimated from above by 2. Finally, the term I_1 is estimated as follows:

$$I_1 \le \frac{1}{96}\max\left\{h_1^2,h_2^2\right\}\int\limits_{|\mathbf{y}|<1}d\mathbf{y} = \frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}. \tag{7}$$

To estimate the term $I_2 := \frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|>1,\mathbf{y}\in Q_{h_1,h_2}}\left(\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}\right|$, integration by parts w.r.t. y_1 will be used. Particularly, considering that the integration domain $|\mathbf{y}|>1\wedge\mathbf{y}\in Q_{h_1,h_2}$ is a rectangular domain with a circular whole of radius 1, the integration by parts leads to the following three summands:

$$I_2 \le \frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|=1}-\frac{1}{ix_1}\left(\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}\cos(\vec{n},y_1)d\mathbf{y}\right| + \frac{1}{(2\pi)^2}\left|-\frac{1}{ix_1}\int\limits_{y_2=-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}}\left(\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}\right)_{\Big|y_1=\frac{\pi}{h_1}}e^{-ix_2y_2}\left(e^{-ix_1\frac{\pi}{h_1}}-e^{ix_1\frac{\pi}{h_1}}\right)dy_2\right|$$
$$+\frac{1}{(2\pi)^2}\left|\frac{1}{ix_1}\int\limits_{|\mathbf{y}|>1,\mathbf{y}\in Q_{h_1,h_2}}\left(\frac{2y_1}{|\mathbf{y}|^4}-\frac{2h_1^{-1}\sin(h_1y_1)}{d^4}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}\right|,$$

where $\vec{n}$ denotes the outer unit normal vector, which is related to the unit circle $|\mathbf{y}| = 1$ in our case, and the second summands combines terms obtained for $y_1 = -\frac{\pi}{h_1}$ and $y_1 = \frac{\pi}{h_1}$. Estimating first two summands similar to I_1, we get:

$$I_2 \leq \frac{1}{192}\frac{1}{|x_1|}\max\left\{h_1^2, h_2^2\right\}\int\limits_{|\mathbf{y}|=1} d\mathbf{y} + \frac{1}{96}\frac{1}{|x_1|}\max\left\{h_1^2, h_2^2\right\}\int\limits_{y_2=-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} dy_2 + \frac{1}{(2\pi)^2}\frac{1}{|x_1|}\int\limits_{|\mathbf{y}|>1, \mathbf{y}\in Q_{h_1,h_2}} \left|\frac{2y_1}{|\mathbf{y}|^4} - \frac{2h_1^{-1}\sin(h_1y_1)}{d^4}\right| d\mathbf{y}.$$

At first, the expression under the last integral is estimated as follows:

$$\begin{aligned}\left|\frac{2y_1}{|\mathbf{y}|^4} - \frac{2h_1^{-1}\sin(h_1y_1)}{d^4}\right| &\leq \left|\frac{2y_1}{|\mathbf{y}|^4} - \frac{2\sin(h_1y_1)}{h_1|\mathbf{y}|^4}\right| + \left|\frac{2\sin(h_1y_1)}{h_1|\mathbf{y}|^4} - \frac{2h_1^{-1}\sin(h_1y_1)}{d^4}\right| \\ &\leq \left|\frac{2y_1 - 2h_1^{-1}\sin(h_1y_1)}{|\mathbf{y}|^4}\right| + \frac{2}{h_1}\left|\sin(h_1y_1)\left(\frac{d^4 - |\mathbf{y}|^4}{d^4|\mathbf{y}|^4}\right)\right| = \mathbf{I} + \mathbf{II}.\end{aligned}$$

Next, expanding $\sin(h_1y_1)$ into Taylor series the term $\mathbf{I}$ is estimated as follows:

$$\mathbf{I} = \left|\frac{2y_1 - 2h_1^{-1}\sin(h_1y_1)}{|\mathbf{y}|^4}\right| = \left|\frac{2y_1 - 2y_1 + 2\frac{h_1^2\cos(h_1y_1\Theta)}{3!}y_1^3}{|\mathbf{y}|^4}\right| = \frac{2h_1^2}{3!}\left|\frac{\cos(h_1y_1\Theta)y_1^3}{|\mathbf{y}|^4}\right| \leq \frac{h_1^2}{3\,|y|}, \text{ with } \Theta \in (0,1).$$

Using the same Taylor expansion for the term $\mathbf{II}$ leads to

$$\mathbf{II} = \frac{2}{h_1}\left|\sin(h_1y_1)\left(\frac{d^4 - |\mathbf{y}|^4}{d^4|\mathbf{y}|^4}\right)\right| \leq \frac{2}{h_1}\left|h_1y_1 - \frac{h_1^3\cos(h_1y_1\Theta)}{3!}y_1^3\right|\left|\frac{d^2 - |\mathbf{y}|^2}{d^2|\mathbf{y}|^2}\right|\left|\frac{d^2 + |\mathbf{y}|^2}{d^2|\mathbf{y}|^2}\right|.$$

The last two factors can be straightforwardly estimated as follows:

$$\left|\frac{d^2 - |\mathbf{y}|^2}{d^2|\mathbf{y}|^2}\right| \leq \frac{\pi^2\max\left\{h_1^2, h_2^2\right\}}{48}, \quad \left|\frac{d^2 + |\mathbf{y}|^2}{d^2|\mathbf{y}|^2}\right| \leq \frac{\pi^2}{2|\mathbf{y}|^2},$$

where inequality (6) and related results have been used. Thus, the term $\mathbf{II}$ is estimated as follows:

$$\begin{aligned}\mathbf{II} &\leq \left|2y_1 - \frac{h_1^2\cos(h_1y_1\Theta)}{3}y_1^3\right| \cdot \frac{\pi^2\max\left\{h_1^2, h_2^2\right\}}{48} \cdot \frac{\pi^2}{2|\mathbf{y}|^2} \leq \left(2\,|y_1| + \left|\frac{h_1^2\cos(h_1y_1\Theta)}{3}y_1^3\right|\right) \cdot \frac{\pi^4\max\left\{h_1^2, h_2^2\right\}}{96|\mathbf{y}|^2} \\ &\leq \left(2\,|\mathbf{y}| + \frac{h_1^2}{3}|\mathbf{y}|^3\right) \cdot \frac{\pi^4\max\left\{h_1^2, h_2^2\right\}}{96|\mathbf{y}|^2} = \frac{\pi^4\max\left\{h_1^2, h_2^2\right\}}{48\,|\mathbf{y}|} + \frac{\pi^4h_1^2\max\left\{h_1^2, h_2^2\right\}}{288}|\mathbf{y}|.\end{aligned}$$

Collecting both estimates for $\mathbf{I}$ and $\mathbf{II}$ leads to

$$\left|\frac{2y_1}{|\mathbf{y}|^4} - \frac{2h_1^{-1}\sin(h_1y_1)}{d^4}\right| \leq \left(\frac{h_1^2}{3} + \frac{\pi^4\max\left\{h_1^2, h_2^2\right\}}{48}\right)\frac{1}{|\mathbf{y}|} + \frac{\pi^4h_1^2\max\left\{h_1^2, h_2^2\right\}}{288}|\mathbf{y}|. \tag{8}$$

Finally, the following estimate for I_2 is obtained:

$$I_2 \leq \frac{\max\left\{h_1^2, h_2^2\right\}}{192 \cdot |x_1|} \int\limits_{|\mathbf{y}|=1} d\mathbf{y} + \frac{\max\left\{h_1^2, h_2^2\right\}}{96 \cdot |x_1|} \int\limits_{y_2=-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} dy_2$$
$$+ \frac{1}{4\pi^2|x_1|} \int\limits_{|\mathbf{y}|>1, \mathbf{y}\in Q_{h_1,h_2}} \left[\left(\frac{h_1^2}{3} + \frac{\pi^4 \max\left\{h_1^2, h_2^2\right\}}{48}\right)\frac{1}{|\mathbf{y}|} + \frac{\pi^4 h_1^2 \max\left\{h_1^2, h_2^2\right\}}{288}|\mathbf{y}|\right] d\mathbf{y}.$$

The last integral has to be calculated by using polar coordinates. To enable the transformation to polar coordinates, the rectangle Q_{h_1,h_2} has been extended to a square with a side-length equal to the maximum side of the original rectangle. Thus, the transformation to polar coordinates could be performed leading the following calculations for $\frac{1}{|\mathbf{y}|}$:

$$\int\limits_{|\mathbf{y}|>1, \mathbf{y}\in Q_{h_1,h_2}} \frac{1}{|\mathbf{y}|} d\mathbf{y} \leq \int\limits_0^{2\pi} \int\limits_1^{\frac{\sqrt{2}\pi}{\min\{h_1,h_2\}}} \frac{1}{r} \cdot r\, dr\, d\varphi = 2\pi \left(\frac{\sqrt{2}\pi}{\min\{h_1, h_2\}} - 1\right).$$

In order to assure positiveness of the expression $\left(\frac{\sqrt{2}\pi}{\min\{h_1,h_2\}} - 1\right)$, a restriction for stepsizes h_1, h_2 needs to be made. In this case, the last term is positive if $\min\{h_1, h_2\} < \sqrt{2}\pi$. Integrating similarly the term $|\mathbf{y}|$ and collecting all results, finally the following estimate for I_2 is obtained:

$$I_2 \leq \frac{\pi \max\left\{h_1^2, h_2^2\right\}}{96|x_1|} + \frac{\pi \max\left\{h_1^2, h_2^2\right\}}{48 h_2 |x_1|} + \frac{1}{|x_1|}\left[\frac{h_1^2}{3\sqrt{2}\min\{h_1, h_2\}} + \frac{\pi^4 \max\left\{h_1^2, h_2^2\right\}}{48\sqrt{2}\min\{h_1, h_2\}}\right.$$
$$\left. + \frac{\pi^6\sqrt{2}h_1^2 \max\left\{h_1^2, h_2^2\right\}}{864 \min\left\{h_1^3, h_2^3\right\}} - \frac{\pi^3 h_1^2 \max\left\{h_1^2, h_2^2\right\}}{1728} - \frac{h_1^2}{6\pi} - \frac{\pi^3 \max\left\{h_1^2, h_2^2\right\}}{96}\right].$$

For the estimation of the third term $I_3 := \frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|\in\mathbb{R}^2\setminus Q_{h_1,h_2}} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}}{|\mathbf{y}|^2} d\mathbf{y}\right|$, again the integration by parts w.r.t. y_1 is used, and taking into account calculation rules for improper integrals it follows:

$$I_3 \leq \frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|\in\mathbb{R}^2\setminus Q_{h_1,h_2}} \frac{e^{-i(x_1y_1+x_2y_2)}}{y_1^2+y_2^2} dy_1 dy_2\right| \leq \frac{1}{(2\pi)^2}\left|\lim_{b\to\infty}\int\limits_{|\mathbf{y}|=b} -\frac{1}{ix_1}\frac{1}{y_1^2+y_2^2} e^{-ixy}\cos(\vec{n}, y_1) d\mathbf{y}\right|$$
$$+ \frac{1}{(2\pi)^2}\left|\frac{1}{ix_1}\int\limits_{y_2=-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \frac{e^{-ix_2y_2}}{\pi^2 h_1^{-2} + y_2^2}\left(e^{-ix_1\frac{\pi}{h_1}} - e^{ix_1\frac{\pi}{h_1}}\right) dy_2\right| + \frac{1}{(2\pi)^2}\left|-\frac{1}{ix_1}\int\limits_{|\mathbf{y}|\in\mathbb{R}^2\setminus Q_{h_1,h_2}} \frac{2y_1}{|y|^4} e^{-ixy} d\mathbf{y}\right|$$
$$\leq \frac{1}{4\pi^2|x_1|}\lim_{b\to\infty}\int\limits_{|\mathbf{y}|=b} \frac{1}{y_1^2+y_2^2} d\mathbf{y} + \frac{1}{2\pi^2|x_1|}\int\limits_{y_2=-\frac{\pi}{h_2}}^{\frac{\pi}{h_2}} \frac{1}{\pi^2 h_1^{-2} + y_2^2} dy_2 + \frac{1}{2\pi^2|x_1|}\int\limits_{|\mathbf{y}|\in\mathbb{R}^2\setminus Q_{h_1,h_2}} \frac{1}{|\mathbf{y}|^3} d\mathbf{y}.$$

The first improper integral tends to zero for $b \to \infty$, as well as the third integral is zero. Thus, the following estimate for I_3 is obtained:

$$I_3 \leq \frac{h_1}{\pi^3|x_1|}\arctan\left(\frac{h_1}{h_2}\right).$$

Similarly, the use of integration by parts w.r.t. y_2 leads to the following estimates for I_2 and I_3:

$$
\begin{aligned}
I_2 \leq\ & \frac{\pi \max\left\{h_1^2, h_2^2\right\}}{96|x_2|} + \frac{\pi \max\left\{h_1^2, h_2^2\right\}}{48h_1|x_2|} + \frac{1}{|x_2|}\left[\frac{h_2^2}{3\sqrt{2}\min\{h_1,h_2\}} + \frac{\pi^4\max\left\{h_1^2,h_2^2\right\}}{48\sqrt{2}\min\{h_1,h_2\}}\right. \\
& \left. + \frac{\pi^6\sqrt{2}h_2^2\max\left\{h_1^2,h_2^2\right\}}{864\min\left\{h_1^3,h_2^3\right\}} - \frac{\pi^3 h_2^2\max\left\{h_1^2,h_2^2\right\}}{1728} - \frac{h_2^2}{6\pi} - \frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}\right], \\
I_3 \leq\ & \frac{h_2}{\pi^3|x_2|}\arctan\left(\frac{h_2}{h_1}\right),
\end{aligned}
$$

where the restriction $\min\{h_1,h_2\} < \sqrt{2}\pi$ has been made again during estimation of I_2.

To obtain the final estimates for I_2 and I_3, the expression $(|x_1|I_k)^2 + (|x_2|I_k)^2$ for $k = 2, 3$ needs to be studied. For $k = 2$ it leads to:

$$
\begin{aligned}
(|x_1|I_2)^2 + (|x_2|I_2)^2 \leq & \left(\frac{\pi \max\left\{h_1^2, h_2^2\right\}}{96} + \frac{\pi \max\left\{h_1^2, h_2^2\right\}}{48h_2} + \frac{h_1^2}{3\sqrt{2}\min\{h_1,h_2\}} + \frac{\pi^4\max\left\{h_1^2,h_2^2\right\}}{48\sqrt{2}\min\{h_1,h_2\}} - \frac{h_1^2}{6\pi}\right. \\
& \left. + \frac{\pi^6\sqrt{2}h_1^2\max\left\{h_1^2,h_2^2\right\}}{864\min\left\{h_1^3,h_2^3\right\}} - \frac{\pi^3 h_1^2\max\left\{h_1^2,h_2^2\right\}}{1728} - \frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}\right)^2 + \left(\frac{\pi \max\left\{h_1^2, h_2^2\right\}}{96} + \frac{h_2^2}{3\sqrt{2}\min\{h_1,h_2\}}\right. \\
& \left. + \frac{\pi \max\left\{h_1^2, h_2^2\right\}}{48h_1} + \frac{\pi^4\max\left\{h_1^2,h_2^2\right\}}{48\sqrt{2}\min\{h_1,h_2\}} + \frac{\pi^6\sqrt{2}h_2^2\max\left\{h_1^2,h_2^2\right\}}{864\min\left\{h_1^3,h_2^3\right\}} - \frac{\pi^3 h_2^2\max\left\{h_1^2,h_2^2\right\}}{1728} - \frac{h_2^2}{6\pi} - \frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}\right)^2 := \mathbf{I}_2,
\end{aligned}
$$

and thus the final estimate for I_2 is obtained

$$
I_2 \leq \frac{1}{|\mathbf{x}|}\sqrt{\mathbf{I}_2}. \tag{9}
$$

Analogously, for $k = 3$ it leads to

$$
(|x_1|I_3)^2 + (|x_2|I_3)^2 \leq \frac{1}{\pi^6}h_1^2\arctan^2\left(\frac{h_1}{h_2}\right) + \frac{1}{\pi^6}h_2^2\arctan^2\left(\frac{h_2}{h_1}\right),
$$

and thus the final estimate for I_3 is obtained:

$$
I_3 \leq \frac{1}{|\mathbf{x}|}\frac{1}{\pi^3}\left[h_1^2\arctan^2\left(\frac{h_1}{h_2}\right) + h_2^2\arctan^2\left(\frac{h_2}{h_1}\right)\right]^{\frac{1}{2}}. \tag{10}
$$

Finally, combining the estimates (7), and (9) and (10) for I_1, I_2 and I_3 the final estimate is obtained as follows:

$$
\begin{aligned}
I_1 + I_2 + I_3 \leq\ & \frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\} + \frac{1}{|\mathbf{x}|}\frac{1}{\pi^3}\left[h_1^2\arctan^2\left(\frac{h_1}{h_2}\right) + h_2^2\arctan^2\left(\frac{h_2}{h_1}\right)\right]^{\frac{1}{2}} \\
& + \frac{1}{|\mathbf{x}|}\left[\left(\frac{\pi \max\left\{h_1^2, h_2^2\right\}}{96} + \frac{\pi \max\left\{h_1^2, h_2^2\right\}}{48h_2} + \frac{h_1^2}{3\sqrt{2}\min\{h_1,h_2\}} + \frac{\pi^4\max\left\{h_1^2,h_2^2\right\}}{48\sqrt{2}\min\{h_1,h_2\}} + \frac{\pi^6\sqrt{2}h_1^2\max\left\{h_1^2,h_2^2\right\}}{864\min\left\{h_1^3,h_2^3\right\}}\right.\right. \\
& \left. - \frac{\pi^3 h_1^2\max\left\{h_1^2,h_2^2\right\}}{1728} - \frac{h_1^2}{6\pi} - \frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}\right)^2 + \left(\frac{\pi \max\left\{h_1^2, h_2^2\right\}}{96} + \frac{\pi \max\left\{h_1^2, h_2^2\right\}}{48h_1} + \frac{h_2^2}{3\sqrt{2}\min\{h_1,h_2\}}\right. \\
& \left.\left. + \frac{\pi^4\max\left\{h_1^2,h_2^2\right\}}{48\sqrt{2}\min\{h_1,h_2\}} + \frac{\pi^6\sqrt{2}h_2^2\max\left\{h_1^2,h_2^2\right\}}{864\min\left\{h_1^3,h_2^3\right\}} - \frac{\pi^3 h_2^2\max\left\{h_1^2,h_2^2\right\}}{1728} - \frac{h_2^2}{6\pi} - \frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}\right)^2\right]^{\frac{1}{2}}.
\end{aligned}
$$

Further, by using estimates $h_1 < \max\{h_1, h_2\}$ and $h_2 < \max\{h_1, h_2\}$ in the numerator of (9), as well as using estimates $h_1 < \min\{h_1, h_2\}$ and $h_2 < \min\{h_1, h_2\}$ in the denominator of (9), and omitting fourth-order term, the above estimate can be finally simplified to the following form

$$I_1 + I_2 + I_3 \leq C_1 \max\left\{h_1^2, h_2^2\right\} + \frac{C_2}{|\mathbf{x}|} \max\{h_1, h_2\} + \frac{C_3}{|\mathbf{x}|} \frac{\max\left\{h_1^2, h_2^2\right\}}{\min\{h_1, h_2\}},$$

where $C_1,\ C_2,\ C_3$ are constants independent on the stepsizes h_1 and h_2. Thus, the assertion of the theorem is proved. □

For a better presentation of the estimate in Theorem 1, the estimate is calculated along different lines of a rectangular lattice. To provide a better overview of the estimate, all plots are calculated for the complete form of the estimate obtain on the pre-last step of the proof of Theorem 1, i.e., without involving extra assumptions for simplification of the final form. Moreover, the influence of ration $\alpha = \frac{h_2}{h_1}$ on the estimate is analysed. Additionally, since the estimates tend asymptotically to zero, only the region with indices till 20 is considered. Figure 1: estimate calculated along coordinate axes and along the main diagonal of the rectangular lattice, i.e., for points $(m_1h_1, 0)$, $(0, m_2h_2)$, and (m_1h_1, m_1h_2), respectively, for $h_1 = \frac{1}{2}$ and $h_2 = \frac{1}{4}$. Figure 2: estimate calculated along the main diagonal of the lattice for different values of ratio α.

As it can be seen from Figures 1 and 2, higher ratio between stepsizes h_1 and h_2 leads to a bigger error; and the lowest error is obtained in the case of $h_1 = h_2$. This behaviour is not surprising, because a square lattice is, in fact, the ideal mesh

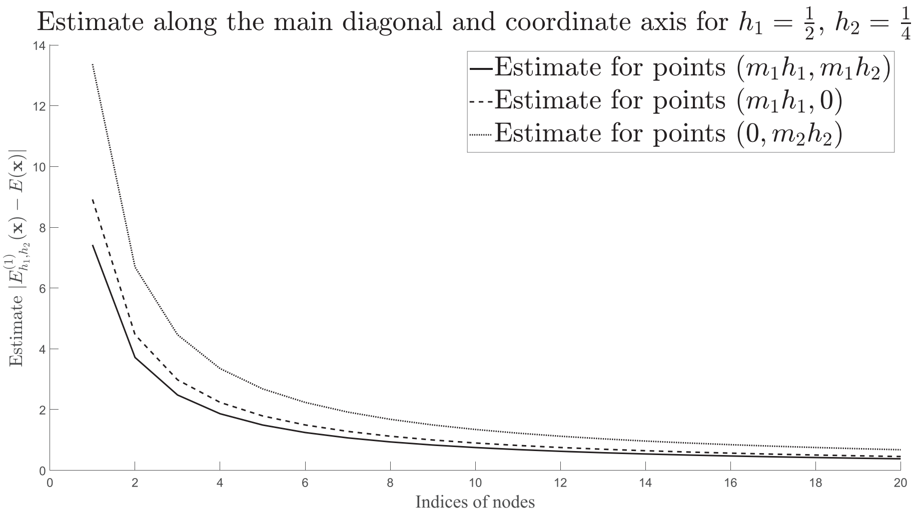

FIGURE 1 Calculation of the error estimate along the main diagonal and coordinate axes based on Theorem 1

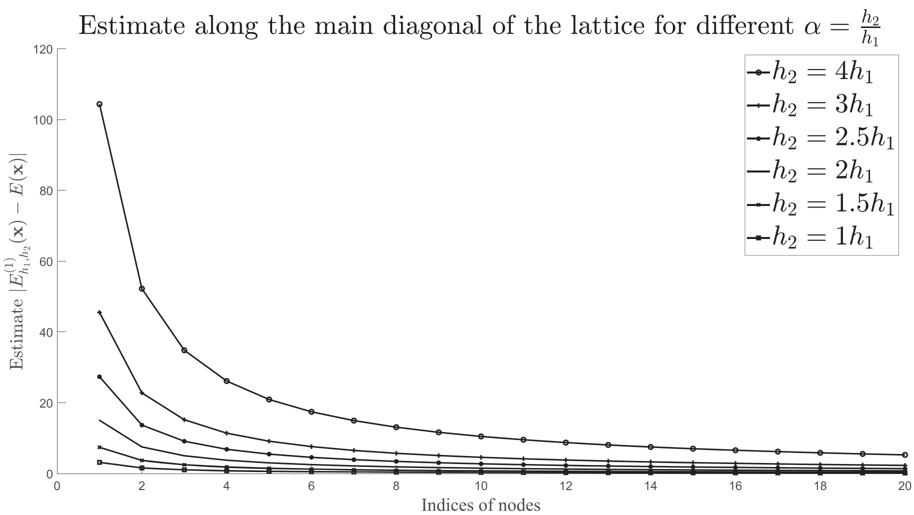

FIGURE 2 Calculation of the error estimate along the main diagonal for different values of α based on Theorem 1

from the point of view of numerical approximation. It is also known, that any deviation from the ideal mesh produces higher approximation error,[14] although providing higher flexibility in practical applications.

For convenience reasons of some theoretical constructions, it is worth to present the following shorter version of the estimate from Theorem 1:

Corollary 1. *Under assumptions of Theorem 1 and assuming $h_2 = \alpha h_1$ for $\alpha \in \mathbb{R}$, the following two cases hold:*

(i) for $\alpha \in (0,1)$: $|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| \leq C_1 h_1^2 + \frac{h_1}{|\mathbf{x}|}\left(C_2 + \frac{C_3}{\alpha}\right)$,
(ii) for $\alpha \in (1,\infty)$: $|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| \leq C_1 \alpha^2 h_1^2 + \frac{\alpha h_1}{|\mathbf{x}|}(C_2 + C_3\alpha)$,

where C_1, C_2, and C_3 are arbitrary constants independent on the stepsizes h_1, h_2, and parameter α.

Analysing the above estimates, it is clear that estimates diverge for $\alpha \to 0$ and $\alpha \to \infty$ for the first and the second case, respectively. This fact is natural because $\alpha \to 0$ and $\alpha \to \infty$ represent extreme cases of a rectangular lattice with infinitely large rectangles in x_1 or x_2 directions. In practice however, the parameter α will always remain finite and positive, and thus, the estimate will always be finite, but can be arbitrary large. Finally, the two above cases can be combined as follows:

Corollary 2. *Under assumptions of Theorem 1 and assuming $h_2 = \alpha h_1$ for $\alpha \in \mathbb{R}$, the following estimate holds:*

$$|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| \leq C_1(\alpha) h_1^2 + C_2(\alpha)\frac{h_1}{|\mathbf{x}|},$$

where $C_1(\alpha)$ and $C_2(\alpha)$ are constants depending on α, which might tend to infinity for $\alpha \to 0$ and $\alpha \to \infty$.

Corollary 2 presents similar behaviour of the estimate (sum of linear and quadratic terms with respect to the stepsize) as in the case of a square lattice.[1,15] However, in the case of a rectangular lattice, only stepsize h_1 appears explicitly in the estimate, while the influence of stepsize h_2 is controlled by α-dependent constants.

Next step is to construct the estimate in $l^p(\Omega_{h_1,h_2})$, where Ω_{h_1,h_2} is a discrete domain:[9]

Theorem 2. *Let $E^{(1)}_{h_1,h_2}$ be the discrete fundamental solution given in (4) of the discrete Laplace operator, and let E be the continuous fundamental solution (3) of the classical Laplace operator. Further let $A(\Omega_{h_1,h_2}) := \sum_{\mathbf{x}\in\Omega_{h_1,h_2}} h_1 h_2$, and let $L_1 := \operatorname{diam}_{x_1} \Omega_{h_1,h_2}$, $L_2 := \operatorname{diam}_{x_2} \Omega_{h_1,h_2}$, i.e., the diameters of Ω_{h_1,h_2} along x_1 and x_2 directions, respectively. Then for all $\mathbf{x} \neq 0$ and $h_1, h_2 < \sqrt{2}\pi$ the following estimates in $l^p(\Omega_{h_1,h_2})$ hold:*

- *for $p = 1$:*

$$\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^1} \leq \frac{\pi}{96}\max\left\{h_1^2, h_2^2\right\} A(\Omega_{h_1,h_2}) + (C_1 + 4\max\{h_1,h_2\}) \times \left(C_2 \max\{h_1,h_2\} + C_3\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}}\right);$$

- *for $1 < p < 2$:*

$$\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^p} \leq \frac{\pi}{96}\max\left\{h_1^2, h_2^2\right\}\left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}} + \left(C_1\max\{h_1,h_2\} + C_2\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}}\right)$$
$$\times\left[\frac{4h_1h_2}{(h_1^2+h_2^2)^{\frac{p}{2}}} + \frac{2h_2}{p-1}\left(2h_1^{1-p} + ph_1^{1-p} - \frac{3}{L_1^{p-1}}\right) + \frac{2h_1}{p-1}\left(2h_2^{1-p} + ph_2^{1-p} - \frac{3}{L_2^{p-1}}\right)\right.$$
$$\left. + \frac{2\pi}{2-p}\left(\left(\sqrt{2}(\max\{L_1,L_2\} - \min\{h_1,h_2\})\right)^{2-p} - (\min\{h_1,h_2\})^{2-p}\right)\right]^{\frac{1}{p}};$$

- *for* $p = 2$*:*

$$\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^2} \leq \frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}\left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{2}} + \left(C_1\max\{h_1,h_2\} + C_2\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}}\right)$$
$$\times\left[\frac{4h_1h_2}{h_1^2+h_2^2} + C_3 - C_4(h_1+h_2) + 2\pi\ln\left|\frac{\sqrt{2}(\max\{L_1,L_2\} - \min\{h_1,h_2\})}{\min\{h_1,h_2\}}\right|\right]^{\frac{1}{2}};$$

- *for* $2 < p < \infty$*:*

$$\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^p} \leq \frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}\left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}} + \left(C_1\max\{h_1,h_2\} + C_2\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}}\right)$$
$$\times\left[\frac{4h_1h_2}{(h_1^2+h_2^2)^{\frac{p}{2}}} + \frac{2h_2}{p-1}\left(2h_1^{1-p} + ph_1^{1-p} - \frac{3}{L_1^{p-1}}\right) + \frac{2h_1}{p-1}\left(2h_2^{1-p} + ph_2^{1-p} - \frac{3}{L_2^{p-1}}\right)\right.$$
$$\left.+\frac{2\pi}{p-2}\left((\min\{h_1,h_2\})^{2-p} - \left(\sqrt{2}(\max\{L_1,L_2\} - \min\{h_1,h_2\})\right)^{2-p}\right)\right]^{\frac{1}{p}};$$

- *for* $p = \infty$*:*

$$\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\|_{l^\infty(\Omega_{h_1,h_2})} \leq \frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\} + \frac{1}{\sqrt{\min\left\{h_1^2,h_2^2\right\}}} \times\left(C_1\max\{h_1,h_2\} + C_2\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}}\right),$$

where all constants are independent on the stepsizes h_1 *and* h_2*.*

Proof. For the sake of completeness, the proof of $l^p(\Omega_{h_1,h_2})$-estimate will be carried out with all long expressions, and it will be simplified to the form presented in the statement of the theorem as the last step of the proof, similarly as it has been done in the proof of Theorem 1. Moreover, for shortening the subscripts, the notation l^p will be used instead of $l^p(\Omega_{h_1,h_2})$. The proof starts with using of the definition of the l^p-norm and applying the Minkowski inequality. After that, considering the proof of Theorem 1, the following expression is obtained:

$$\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^p} = \left(\sum_{(m_1h_1,m_2h_2)\in\Omega_{h_1,h_2}}\left|E^{(1)}_{h_1,h_2}(m_1h_1,m_2h_2) - E(m_1h_1,m_2h_2)\right|^p h_1h_2\right)^{\frac{1}{p}}$$
$$\leq \left\|\frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}\right\|_{l^p} + \left\|\frac{1}{|\mathbf{x}|}\frac{1}{\pi^3}\left(h_1^2\arctan^2\left(\frac{h_1}{h_2}\right) + h_2^2\arctan^2\left(\frac{h_2}{h_1}\right)\right)^{\frac{1}{2}}\right\|_{l^p}$$
$$+\left\|\frac{1}{|\mathbf{x}|}\left[\left(\frac{\pi\max\left\{h_1^2,h_2^2\right\}}{96} + \frac{\pi\max\left\{h_1^2,h_2^2\right\}}{48h_2} + \frac{h_1^2}{3\sqrt{2}\min\{h_1,h_2\}} + \frac{\pi^4\max\left\{h_1^2,h_2^2\right\}}{48\sqrt{2}\min\{h_1,h_2\}} + \frac{\pi^6\sqrt{2}h_1^2\max\left\{h_1^2,h_2^2\right\}}{864\min\left\{h_1^3,h_2^3\right\}}\right.\right.\right.$$
$$\left.-\frac{\pi^3h_1^2\max\left\{h_1^2,h_2^2\right\}}{1728} - \frac{h_1^2}{6\pi} - \frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}\right)^2 + \left(\frac{\pi\max\left\{h_1^2,h_2^2\right\}}{96} + \frac{\pi\max\left\{h_1^2,h_2^2\right\}}{48h_1} + \frac{h_2^2}{3\sqrt{2}\min\{h_1,h_2\}}\right.$$
$$\left.\left.\left.+\frac{\pi^4\max\left\{h_1^2,h_2^2\right\}}{48\sqrt{2}\min\{h_1,h_2\}} + \frac{\pi^6\sqrt{2}h_2^2\max\left\{h_1^2,h_2^2\right\}}{864\min\left\{h_1^3,h_2^3\right\}} - \frac{\pi^3h_2^2\max\left\{h_1^2,h_2^2\right\}}{1728} - \frac{h_2^2}{6\pi} - \frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}\right)^2\right]^{\frac{1}{2}}\right\|_{l^p}.$$

After shorting notations by defining $A(\Omega_{h_1,h_2}) := \sum\limits_{\mathbf{x}\in\Omega_{h_1,h_2}} h_1h_2$, the following estimate can be obtained:

$$
\begin{aligned}
&\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^p} \leq \frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}\left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}} + \left\|\frac{1}{|\mathbf{x}|}\right\|_{l^p}\frac{1}{\pi^3}\left(h_1^2\arctan^2\left(\frac{h_1}{h_2}\right)+h_2^2\arctan^2\left(\frac{h_2}{h_1}\right)\right)^{\frac{1}{2}} \\
&+\left\|\frac{1}{|\mathbf{x}|}\right\|_{l^p}\left[\left(\frac{\pi\max\left\{h_1^2,h_2^2\right\}}{96}+\frac{\pi\max\left\{h_1^2,h_2^2\right\}}{48h_2}+\frac{h_1^2}{3\sqrt{2}\min\{h_1,h_2\}}+\frac{\pi^4\max\left\{h_1^2,h_2^2\right\}}{48\sqrt{2}\min\{h_1,h_2\}}+\frac{\pi^6\sqrt{2}h_1^2\max\left\{h_1^2,h_2^2\right\}}{864\min\left\{h_1^3,h_2^3\right\}}\right.\right. \\
&\left.-\frac{\pi^3h_1^2\max\left\{h_1^2,h_2^2\right\}}{1728}-\frac{h_1^2}{6\pi}-\frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}\right)^2+\left(\frac{\pi\max\left\{h_1^2,h_2^2\right\}}{96}+\frac{\pi\max\left\{h_1^2,h_2^2\right\}}{48h_1}+\frac{h_2^2}{3\sqrt{2}\min\{h_1,h_2\}}\right. \\
&\left.\left.+\frac{\pi^4\max\left\{h_1^2,h_2^2\right\}}{48\sqrt{2}\min\{h_1,h_2\}}+\frac{\pi^6\sqrt{2}h_2^2\max\left\{h_1^2,h_2^2\right\}}{864\min\left\{h_1^3,h_2^3\right\}}-\frac{\pi^3h_2^2\max\left\{h_1^2,h_2^2\right\}}{1728}-\frac{h_2^2}{6\pi}-\frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}\right)^2\right]^{\frac{1}{2}}.
\end{aligned}
$$

By using the same simplification ideas as during the proof of Theorem 1, the above estimate can be reduced to the form:

$$
\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^p} \leq \frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}\left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}} + \left\|\frac{1}{|\mathbf{x}|}\right\|_{l^p}\left(C_1\max\{h_1,h_2\}+C_2\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}}\right).
$$

Application of the definition of the l^p-norm to the term $\frac{1}{|\mathbf{x}|}$ leads to the following expression

$$
\left\|\frac{1}{|\mathbf{x}|}\right\|_{l^p} = \left(4\sum_{m_1=1}^{l_1}\sum_{m_2=1}^{l_2}\frac{h_1h_2}{(m_1^2h_1^2+m_2^2h_2^2)^{\frac{p}{2}}}+2\sum_{m_1=1}^{l_1}\frac{h_1h_2}{(m_1h_1)^p}+2\sum_{m_2=1}^{l_2}\frac{h_1h_2}{(m_2h_2)^p}\right)^{\frac{1}{p}}, \tag{11}
$$

where l_1 and l_2 denote maximal indices of a domain Ω_{h_1,h_2} in x_1 and x_2 directions correspondingly. Note that the indices l_1 and l_2 depend on stepsizes h_1 and h_2, and, in fact, they are inversely proportional to h_1 and h_2, respectively. Keeping this information aside, the proof will be constructed, and during final simplifications at the very end of the proof, the dependencies of indices on stepsizes will be addressed.

Since the functions under summations in (11) are monotone decreasing functions, the estimation of these sums will be based on the integral test. Each sum will be estimated individually. Since the single sums can be estimated easily, at first the second and the third sums will be considered. The following estimates are obtained for the second and the third sum:

$$
\sum_{m_1=1}^{l_1}\frac{h_1h_2}{(m_1h_1)^p} \leq \frac{h_1h_2}{h_1^p}+\int\limits_1^{l_1}\frac{h_1h_2}{(m_1h_1)^p}dm_1 \leq \begin{cases} h_2(1+\ln|l_1|), & p=1, \\ h_1^{1-p}h_2\frac{p-l_1^{1-p}}{p-1}, & p>1, \end{cases} \quad \sum_{m_2=1}^{l_2}\frac{h_1h_2}{(m_2h_2)^p} \leq \begin{cases} h_1(1+\ln|l_2|), & p=1, \\ h_1h_2^{1-p}\frac{p-l_2^{1-p}}{p-1}, & p>1. \end{cases}
$$

Application of the integral test to the double sum leads to the following estimate:

$$
\sum_{m_1=1}^{l_1}\sum_{m_2=1}^{l_2}\frac{h_1h_2}{(m_1^2h_1^2+m_2^2h_2^2)^{\frac{p}{2}}} \leq \frac{h_1h_2}{(h_1^2+h_2^2)^{\frac{p}{2}}}+\int\limits_{h_1}^{l_1h_1}\frac{h_2}{(x^2+h_2^2)^{\frac{p}{2}}}dx+\int\limits_{h_2}^{l_2h_2}\frac{h_1}{(h_1^2+y^2)^{\frac{p}{2}}}dy+\int\limits_{h_1}^{l_1h_1}\int\limits_{h_2}^{l_2h_2}\frac{1}{(x^2+y^2)^{\frac{p}{2}}}dydx. \tag{12}
$$

The technique to estimate double sum (12) depends on the number p. Therefore, at first the case $p = 1$ will be considered, because in that case the double sum can be estimated again by help of the integral test explicitly as follows:

$$\sum_{m_1=1}^{l_1}\sum_{m_2=1}^{l_2}\frac{h_1h_2}{(m_1^2h_1^2+m_2^2h_2^2)^{\frac{p}{2}}}\leq\frac{h_1h_2}{\sqrt{h_1^2+h_2^2}}+l_1h_1\ln\left|\frac{l_2+\sqrt{l_2^2+\frac{l_1^2h_1^2}{h_2^2}}}{1+\sqrt{1+\frac{l_1^2h_1^2}{h_2^2}}}\right|+l_2h_2\ln\left|\frac{l_1+\sqrt{l_1^2+\frac{l_2^2h_2^2}{h_1^2}}}{1+\sqrt{1+\frac{l_2^2h_2^2}{h_1^2}}}\right|.$$

The estimation of the double integral in (12) for a general p is more difficult, since only for integer values of p this integral can be calculated explicitly, and even in that case, an iterative application of known integrals is needed.[16] Therefore, instead of estimating the double integral over a rectangular domain, the rectangular domain is extended to the biggest possible square constructed based on side-lengths of the rectangular, similar to the idea used during the proof of Theorem 1. Thus, the double integral over a rectangular domain is estimated by a double integral over the biggest square as follows:

$$\int_{h_1}^{l_1h_1}\int_{h_2}^{l_2h_2}\frac{1}{(x^2+y^2)^{\frac{p}{2}}}dydx\leq\int_{\min\{h_1,h_2\}}^{\max\{l_1h_1,l_2h_2\}}\int_{\min\{h_1,h_2\}}^{\max\{l_1h_1,l_2h_2\}}\frac{1}{(x^2+y^2)^{\frac{p}{2}}}dydx.$$

The construction, proposed above, enables the use of polar coordinates for an exact calculation of the double integral over the biggest square. Thus, the integral can be estimated now as follows:

$$\int_{\min\{h_1,h_2\}}^{\max\{l_1h_1,l_2h_2\}}\int_{\min\{h_1,h_2\}}^{\max\{l_1h_1,l_2h_2\}}\frac{1}{(x^2+y^2)^{\frac{p}{2}}}dydx\leq\int_{\min\{h_1,h_2\}}^{\sqrt{2}(\max\{l_1h_1,l_2h_2\}-\min\{h_1,h_2\})}\int_0^{\frac{\pi}{2}}\frac{1}{r^p}rd\varphi dr=$$

$$=\begin{cases}\frac{\pi}{2}\ln\left|\frac{\sqrt{2}((\max\{l_1h_1,l_2h_2\}-\min\{h_1,h_2\}))}{\min\{h_1,h_2\}}\right|, & p=2,\\ \frac{\pi}{2(2-p)}\left[\sqrt{2}(\max\{l_1h_1,l_2h_2\}-\min\{h_1,h_2\})^{2-p}-(\min\{h_1,h_2\})^{2-p}\right], & p\neq 2.\end{cases}$$

The one-dimensional integrals in (12) for $p>1$ and $p\neq 2$ can be estimated as follows:

$$\int_{h_1}^{l_1h_1}\frac{h_2}{(x^2+h_2^2)^{\frac{p}{2}}}dx\leq\int_{h_1}^{l_1h_1}\frac{h_2}{x^p}dx\leq\frac{1}{1-p}h_2h_1^{1-p}(l_1^{1-p}-1),\quad\int_{h_2}^{l_2h_2}\frac{h_1}{(h_1^2+y^2)^{\frac{p}{2}}}dy\leq\int_{h_2}^{l_2h_2}\frac{h_1}{y^p}dy\leq\frac{1}{1-p}h_1h_2^{1-p}(l_2^{1-p}-1).$$

Finally, for $p=2$ the double sum can be estimated as follows:

$$\sum_{m_1=1}^{l_1}\sum_{m_2=1}^{l_2}\frac{h_1h_2}{(m_1^2h_1^2+m_2^2h_2^2)^{\frac{p}{2}}}\leq\frac{h_1h_2}{h_1^2+h_2^2}+\arctan\left(\frac{l_1h_1}{h_2}\right)-\arctan\left(\frac{h_1}{h_2}\right)$$
$$+\arctan\left(\frac{l_2h_2}{h_1}\right)-\arctan\left(\frac{h_2}{h_1}\right)+\frac{\pi}{2}\ln\left|\frac{\sqrt{2}((\max\{l_1h_1,l_2h_2\}-\min\{h_1,h_2\}))}{\min\{h_1,h_2\}}\right|.$$

Summarising the above results for different values of p, the following estimates are obtained:

- for $p = 1$:

$$\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^1} \leq \frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}A(\Omega_{h_1,h_2}) + \left(C_1\max\{h_1,h_2\} + C_2\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}}\right)$$
$$\times\left(4l_1h_1\ln\left|\frac{l_2+\sqrt{l_2^2+\frac{l_1^2h_1^2}{h_2^2}}}{1+\sqrt{1+\frac{l_1^2h_1^2}{h_2^2}}}\right| + \frac{4h_1h_2}{\sqrt{h_1^2+h_2^2}} + 2h_2(1+\ln|l_1|) + 4l_2h_2\ln\left|\frac{l_1+\sqrt{l_1^2+\frac{l_2^2h_2^2}{h_1^2}}}{1+\sqrt{1+\frac{l_2^2h_2^2}{h_1^2}}}\right| + 2h_1(1+\ln|l_2|)\right);$$

- for $1 < p < 2$:

$$\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^p} \leq \frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}\left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}} + \left(C_1\max\{h_1,h_2\} + C_2\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}}\right)$$
$$\times\left[\frac{4h_1h_2}{(h_1^2+h_2^2)^{\frac{p}{2}}} + \frac{2h_1^{1-p}h_2}{p-1}\left(2+p-\frac{3}{l_1^{p-1}}\right) + \frac{2h_1h_2^{1-p}}{p-1}\left(2+p-\frac{3}{l_2^{p-1}}\right)\right.$$
$$\left.+\frac{2\pi}{2-p}\left(\left(\sqrt{2}(\max\{l_1h_1,l_2h_2\}-\min\{h_1,h_2\})\right)^{2-p} - (\min\{h_1,h_2\})^{2-p}\right)\right]^{\frac{1}{p}};$$

- for $p = 2$:

$$\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^2} \leq \frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}\left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{2}} + \left(C_1\max\{h_1,h_2\} + C_2\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}}\right)$$
$$\times\left[\frac{4h_1h_2}{h_1^2+h_2^2} + 4\arctan\left(\frac{l_1h_1}{h_2}\right) - 4\arctan\left(\frac{h_1}{h_2}\right) + 4\arctan\left(\frac{l_2h_2}{h_1}\right) - 4\arctan\left(\frac{h_2}{h_1}\right) + 2\frac{h_2}{h_1}\cdot\frac{2l_1-1}{l_1}\right.$$
$$\left.+2\frac{h_1}{h_2}\cdot\frac{2l_2-1}{l_2} + 2\pi\ln\left|\frac{\sqrt{2}(\max\{l_1h_1,l_2h_2\}-\min\{h_1,h_2\})}{\min\{h_1,h_2\}}\right|\right]^{\frac{1}{2}};$$

- for $2 < p < \infty$:

$$\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^p} \leq \frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\}\left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}} + \left(C_1\max\{h_1,h_2\} + C_2\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}}\right)$$
$$\times\left[\frac{4h_1h_2}{(h_1^2+h_2^2)^{\frac{p}{2}}} + \frac{2h_1^{1-p}h_2}{p-1}\left(2+p-\frac{3}{l_1^{p-1}}\right) + \frac{2h_1h_2^{1-p}}{p-1}\left(2+p-\frac{3}{l_2^{p-1}}\right)\right.$$
$$\left.+\frac{2\pi}{p-2}\left((\min\{h_1,h_2\})^{2-p} - \left(\sqrt{2}(\max\{l_1h_1,l_2h_2\}-\min\{h_1,h_2\})\right)^{2-p}\right)\right]^{\frac{1}{p}}.$$

Finally, dependencies of the indices l_1 and l_2 on h_1 and h_2 for a fixed domain Ω_{h_1,h_2} needs to be taken into account. To overcome this issue, instead of working with indices of points, the quantities $L_1 = l_1h_1$ and $L_2 = l_2h_2$ representing diameters of Ω_{h_1,h_2} in x_1 and x_2 directions, respectively, will be considered. After that, the above estimates can be simplified to the form presented in the theorem, and the remaining step is to construct l^∞-estimate. Using the definition of the norm leads to:

$$\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\|_{l^\infty(\Omega_{h_1,h_2})} = \sup_{\mathbf{x}\in\Omega_{h_1,h_2}}|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})|,$$

where $\mathbf{x} = (m_1h_1, m_2h_2)$ with $m_1, m_2 \in \mathbb{Z}$, and $h_1,\ h_2$ are stepsizes tending to zero. Recalling the estimate provided by Theorem 1:

$$|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| \leq \frac{\pi}{96}\max\left\{h_1^2,h_2^2\right\} + \frac{1}{|\mathbf{x}|}\left(C_1\max\{h_1,h_2\} + C_2\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}}\right),$$

and taking into account the definition of $|\mathbf{x}|$ and simplifying the resulting expression, the following estimate is obtained:

$$\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\|_{l^\infty(\Omega_{h_1,h_2})} \leq \frac{\pi}{96}\max\left\{h_1^2, h_2^2\right\} + \frac{1}{\sqrt{\min\left\{h_1^2,h_2^2\right\}|m|}} \times \left(C_1 \max\left\{h_1,h_2\right\} + C_2 \frac{\max\left\{h_1^2,h_2^2\right\}}{\min\left\{h_1,h_2\right\}} \right),$$

where $|m| = \sqrt{m_1^2 + m_2^2}$. Finally, noticing that the fraction $\frac{1}{\sqrt{\min\left\{h_1^2,h_2^2\right\}|m|}}$ takes its maximum for $|m| = 1$, the final estimate is obtained. Thus, the theorem is proved. □

To illustrate the l^p estimates for different values of p, the estimates are calculated with $h_2 = \alpha h_1$ for $\alpha = 3$ and decreasing values of h_1. Figure 3 shows calculations the estimates for different values of p with respect to the stepsize h_1 in a logarithmic scale. Similar to the results presented in Figures 1 and 2, the l^p-error is smaller if the parameter α is close to 1. As it can be clearly seen, all estimates converge to zero for $h_1 \to 0$ for $p = [1, \infty)$, as expected. The case $p = \infty$ illustrates as h_1 tends to zero, the l^∞ estimate represents the difference between continuous and discrete fundamental solutions at a point arbitrary close the coordinate origin, where the continuous fundamental solution has singularity, and therefore, this difference cannot become zero for any arbitrary small, but finite, stepsize h_1 (since in this example $h_2 = \alpha h_1$).

Similar to the discussion after Theorem 1 summarised in the form of Corollary 2, it is worth to present the following corollary:

Corollary 3. *Under assumptions of Theorem 2, and assuming $h_2 = \alpha h_1$ for $\alpha \in \mathbb{R}$, then the following estimates hold:*

- *for $p = 1$:* $\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^1} \leq \frac{\pi}{96}\alpha^2 h_1^2 A(\Omega_{h_1,h_2}) + C_1(\alpha)h_1 + C_2(\alpha)h_1^2$;
- *for $1<p<2$:* $\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^p} \leq \frac{\pi}{96}\alpha^2 h_1^2 \left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}} + h_1 C_1(\alpha)\left(C_2(\alpha,p)h_1^{2-p} - C_3(\alpha,p)h_1\right)^{\frac{1}{p}}$;
- *for $p = 2$:* $\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^2} \leq \frac{\pi}{96}\alpha^2 h_1^2 \left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{2}} + h_1 C_1(\alpha)(C_3(\alpha) - C_2(\alpha)h_1 - 2\pi \ln h_1)^{\frac{1}{2}}$;
- *for $2<p<\infty$:* $\left\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\right\|_{l^p} \leq \frac{\pi}{96}\alpha^2 h_1^2 \left[A(\Omega_{h_1,h_2})\right]^{\frac{1}{p}} + h_1 C_1(\alpha)\left(C_2(\alpha,p)h_1^{p-2} - C_3(\alpha,p)h_1\right)^{\frac{1}{p}}$;
- *for $p = \infty$:* $\|E^{(1)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})\|_{l^\infty} \leq C_1(\alpha)h_1^2 + C_2(\alpha)$,

where some of the constants depend on α and p, while the other depend only on α. Moreover, all constant tend to infinity for $\alpha \to 0$ and $\alpha \to \infty$.

3.2 | Estimates for the discrete fundamental solution $E^{(2)}_{h_1,h_2}$

Looking at the l^p-estimates for the discrete fundamental solution $E^{(1)}_{h_1,h_2}$ provided in Theorem 2, it becomes clear that because of the term $A(\Omega_{h_1,h_2})$ a similar estimate cannot be obtained for the exterior domain $\Omega^{ext}_{h_1,h_2}$, since the related sum

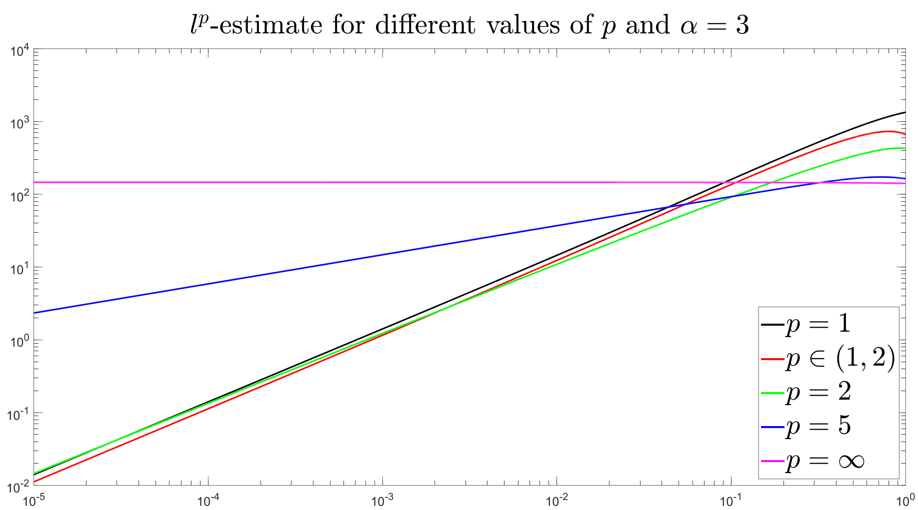

FIGURE 3 Calculation of $l^p(\Omega_{h_1,h_2})$-error estimates from Theorem 2 in a logarithmic scale with respect to h_1 for $h_2 = 3h_1$ for a rectangular domain with length $L_1 = 1$ and height $L_2 = 2$ [Colour figure can be viewed at wileyonlinelibrary.com]

$A(\Omega^{ext}_{h_1,h_2})$ will be a sum over infinite set. To overcome this problem, another regularised version of the discrete fundamental solution is considered:

$$E^{(2)}_{h_1,h_2}(\mathbf{x}) = \frac{1}{(2\pi)^2}\left(\int\limits_{|\mathbf{y}|<1} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}-1}{d^2}d\mathbf{y} + \int\limits_{|\mathbf{y}|>1, y\in Q_{h_1,h_2}} \frac{e^{-i\mathbf{x}\cdot\mathbf{y}}}{d^2}d\mathbf{y} + \int\limits_{|\mathbf{y}|<1}\left(\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}\right)d\mathbf{y}\right), \tag{13}$$

which is more suitable for applications in unbounded domains.[15] The fundamental solution $E^{(2)}_{h_1,h_2}$ is different to the $E^{(1)}_{h_1,h_2}$ by the following constant:

$$K_2 = \frac{1}{(2\pi)^2}\int\limits_{|\mathbf{y}|<1}\left(\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}\right)d\mathbf{y},$$

which depends on $h_1,\ h_2$, since d^2 contains the stepsizes. Moreover, considering that $d^2 \to |\mathbf{y}|^2$ for $h_1 \to 0$ and $h_2 \to 0$, the constant K_2 tends to zero as well. Additionally, consider the difference

$$E^{(2)}_{h_1,h_2}(\mathbf{x}) - E_{h_1,h_2}(\mathbf{x}) = E^{(1)}_{h_1,h_2}(\mathbf{x}) + K_2 - E_{h_1,h_2}(\mathbf{x}) = K_1 + K_2,$$

which states the relation between three different formulations of the discrete fundamental solution of the discrete Laplace operator. Based on previous calculations, the sum $K_1 + K_2$ can be estimated as follows:

$$K_1 + K_2 \le \frac{2\pi \max\left\{h_1^2, h_2^2\right\}}{\min\left\{h_1^2, h_2^2\right\}} \ln\left(\frac{\sqrt{2}\pi}{\min\left\{h_1, h_2\right\}}\right) + \frac{\pi}{192}\max\left\{h_1^2, h_2^2\right\},$$

where each summand corresponds to the estimates for K_1 and K_2, respectively.

Theorem 3. *Let $E^{(2)}_{h_1,h_2}$ be the discrete fundamental solution given in (13) of the discrete Laplace operator, and let E be the continuous fundamental solution (3) of the classical Laplace operator. Then for all $\mathbf{x}\neq 0$ and all $h_1, h_2 < \sqrt{2}\pi$ the following estimate holds*

$$|E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| \le \frac{C_1}{|\mathbf{x}|}\max\left\{h_1, h_2\right\} + \frac{C_2}{|\mathbf{x}|}\frac{\max\left\{h_1^2, h_2^2\right\}}{\min\left\{h_1, h_2\right\}} + \frac{C_3}{|\mathbf{x}|}\max\left\{h_1^2, h_2^2\right\},$$

where the constants $C_1,\ C_2$, and C_3 do not depend on the stepsizes h_1 and h_2.

Proof. Analogously to (5), application of the triangle inequality leads to

$$|E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| \le \frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|<1}\left(\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}\right|$$
$$+\frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|>1, y\in Q_{h_1,h_2}}\left(\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}\right| + \frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|\in\mathbb{R}^2\setminus Q_{h_1,h_2}}\frac{e^{-i\mathbf{x}\cdot\mathbf{y}}}{|\mathbf{y}|^2}d\mathbf{y}\right|.$$

By help of the proof of Theorem 1 the following estimate is obtained on the first step:

$$|E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x})| \le I_2 + I_3 + \frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|<1}\left(\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}\right|,$$

where I_2 and I_3 are terms estimated during the proof of Theorem 1. Thus, the following term must be estimated:

$$I_4 = \frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|<1}\left(\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}\right| = \lim_{\epsilon\to 0}\frac{1}{(2\pi)^2}\left|\int\limits_{|\mathbf{y}|\le\epsilon}\left(\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}+\int\limits_{\epsilon<|\mathbf{y}|<1}\left(\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}\right|$$

By using integration by parts w.r.t. y_1, the estimate of $\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}$ provided in (6) and estimate of $\left|\frac{2y_1}{|\mathbf{y}|^4}-\frac{2h_1^{-1}\sin(h_1y_1)}{d^4}\right|$ given in (8), the following estimate is obtained:

$$\begin{aligned}I_4 &\le \frac{1}{(2\pi)^2}\cdot\frac{\pi^2}{48}\cdot\max\left\{h_1^2,h_2^2\right\}\lim_{\epsilon\to 0}\int\limits_{|\mathbf{y}|\le\epsilon}d\mathbf{y}+\frac{1}{(2\pi)^2}\lim_{\epsilon\to 0}\left|\frac{1}{ix_1}\int\limits_{\epsilon<|\mathbf{y}|<1}\left(\frac{2y_1}{|\mathbf{y}|^4}-\frac{2h_1^{-1}\sin(h_1y_1)}{d^4}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}d\mathbf{y}\right|\\
&+\frac{1}{(2\pi)^2}\lim_{\epsilon\to 0}\left|-\frac{1}{ix_1}\int\limits_{|\mathbf{y}|=\epsilon}\left(\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}\cos(\vec{n},y_1)d\mathbf{y}\right|+\frac{1}{(2\pi)^2}\lim_{\epsilon\to 0}\left|-\frac{1}{ix_1}\int\limits_{|\mathbf{y}|=1}\left(\frac{1}{d^2}-\frac{1}{|\mathbf{y}|^2}\right)e^{-i\mathbf{x}\cdot\mathbf{y}}\cos(\vec{n},y_1)d\mathbf{y}\right|\\
&\le\frac{1}{4\pi^2}\frac{1}{|x_1|}\lim_{\epsilon\to 0}\left(\int\limits_{\epsilon<|\mathbf{y}|<1}\left(\left[\frac{h_1^2}{3}+\frac{\pi^4\max\left\{h_1^2,h_2^2\right\}}{48}\right]\frac{1}{|\mathbf{y}|}+\frac{\pi^4h_1^2\max\left\{h_1^2,h_2^2\right\}}{288}|\mathbf{y}|\right)d\mathbf{y}+\frac{\pi^2\max\left\{h_1^2,h_2^2\right\}}{48}\int\limits_{|\mathbf{y}|=\epsilon}d\mathbf{y}\right.\\
&\left.+\frac{\pi^2\max\left\{h_1^2,h_2^2\right\}}{48}\int\limits_{|\mathbf{y}|=1}d\mathbf{y}\right)\le\frac{1}{|x_1|}\left(\frac{h_1^2}{6\pi}+\frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}+\frac{\pi^3h_1^2\max\left\{h_1^2,h_2^2\right\}}{1728}+\frac{\pi\max\left\{h_1^2,h_2^2\right\}}{96}\right).\end{aligned}$$

Analogously, integration by parts w.r.t. y_2 leads to the estimate:

$$I_4\le\frac{1}{|x_2|}\left(\frac{h_2^2}{6\pi}+\frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}+\frac{\pi^3h_2^2\max\left\{h_1^2,h_2^2\right\}}{1728}+\frac{\pi\max\left\{h_1^2,h_2^2\right\}}{96}\right).$$

Using again the expression $(|x_1|I_4)^2+(|x_2|I_4)^2$, as it has been done in the proof of Theorem 1, finally the estimate for I_4 is obtained in the following form:

$$\begin{aligned}I_4\le\frac{1}{|\mathbf{x}|}&\left[\left(\frac{h_1^2}{6\pi}+\frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}+\frac{\pi^3h_1^2\max\left\{h_1^2,h_2^2\right\}}{1728}+\frac{\pi\max\left\{h_1^2,h_2^2\right\}}{96}\right)^2\right.\\
&\left.+\left(\frac{h_2^2}{6\pi}+\frac{\pi^3\max\left\{h_1^2,h_2^2\right\}}{96}+\frac{\pi^3h_2^2\max\left\{h_1^2,h_2^2\right\}}{1728}+\frac{\pi\max\left\{h_1^2,h_2^2\right\}}{96}\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

Finally, by collecting all terms together and simplifying the estimate for I_4, the theorem is proved. □

Similar analysis, as the one performed for the estimate presented in Theorem 1, is made for the estimate in Theorem 3. The results of analysis are summarised in Figures 4 and 5.

Similar to the discussion around the discrete fundamental solution $E^{(1)}_{h_1,h_2}$, next corollary is introduced:

Corollary 4. *Under assumptions of Theorem 3, let us further assume that* $h_2=\alpha h_1$ *for* $\alpha\in\mathbb{R}$*, then the following estimate holds:*

$$|E^{(2)}_{h_1,h_2}(\mathbf{x})-E(\mathbf{x})|\le\frac{1}{|\mathbf{x}|}\left(C_1(\alpha)h_1+C_2(\alpha)h_1^2\right),$$

where $C_1(\alpha)$ *and* $C_2(\alpha)$ *tend to infinity for* $\alpha\to 0$ *and* $\alpha\to\infty$*.*

Theorem 4. *Let* $E^{(2)}_{h_1,h_2}$ *be the discrete fundamental solution given in (13) of the discrete Laplace operator, and let E be the continuous fundamental solution (3) of the classical Laplace operator. Let* $L_1:=\operatorname{diam}_{x_1}\Omega_{h_1,h_2}$, $L_2:=\operatorname{diam}_{x_2}\Omega_{h_1,h_2}$, *i.e., the*

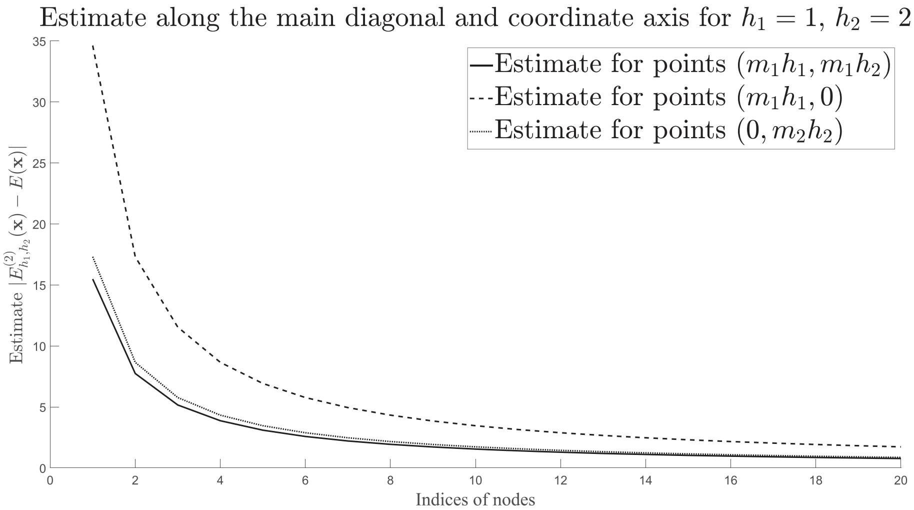

FIGURE 4 Calculation of the error estimate along the main diagonal and coordinate axes based on Theorem 3

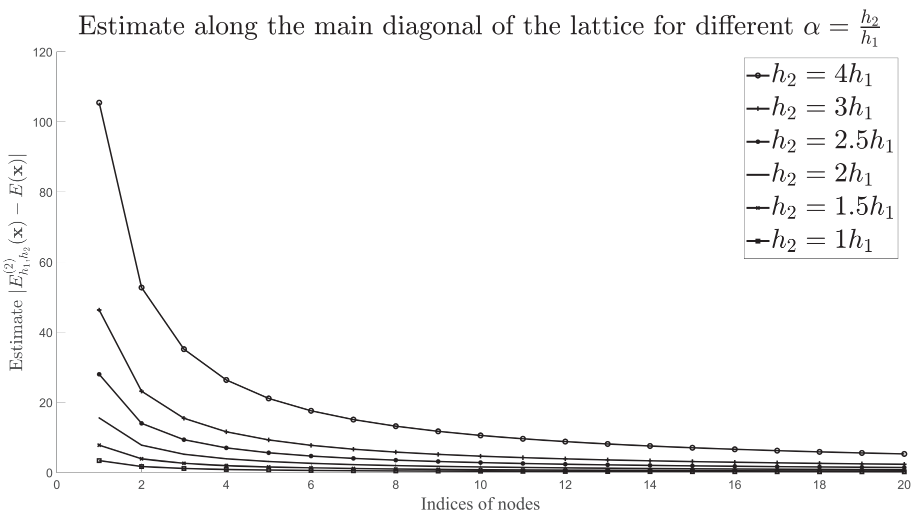

FIGURE 5 Calculation of the error estimate along the main diagonal for different values of α based on Theorem 3

diameters of Ω_{h_1,h_2} along x_1 and x_2 directions, respectively. Then for all $\mathbf{x} \neq 0$ and $h_1, h_2 < \sqrt{2}\pi$ the following estimates in $l^p(\Omega_{h_1,h_2})$ hold:

- *for $p = 1$:*

$$\left\| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^1} \leq (C_1 + 4\max\{h_1,h_2\}) \left(C_2 \max\{h_1,h_2\} + C_3 \frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}} + C_4 \max\{h_1^2,h_2^2\} \right);$$

- *for $1 < p < 2$:*

$$\begin{aligned}\left\| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^p} \leq & \left(C_1 \max\{h_1,h_2\} + C_2 \frac{\max\{h_1^2,h_2^2\}}{\min\{h_1,h_2\}} + C_3 \max\{h_1^2,h_2^2\} \right) \\ & \times \left[\frac{4h_1h_2}{(h_1^2+h_2^2)^{\frac{p}{2}}} + \frac{2h_2}{p-1}\left(2h_1^{1-p} + ph_1^{1-p} - \frac{3}{L_1^{p-1}}\right) + \frac{2h_1}{p-1}\left(2h_2^{1-p} + ph_2^{1-p} - \frac{3}{L_2^{p-1}}\right) \right. \\ & \left. + \frac{2\pi}{2-p}\left(\left(\sqrt{2}(\max\{L_1,L_2\} - \min\{h_1,h_2\})\right)^{2-p} - (\min\{h_1,h_2\})^{2-p} \right) \right]^{\frac{1}{p}};\end{aligned}$$

- *for* $p = 2$*:*

$$\left\| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^2} \leq \left(C_1 \max\{h_1, h_2\} + C_2 \frac{\max\left\{h_1^2, h_2^2\right\}}{\min\{h_1, h_2\}} + C_3 \max\left\{h_1^2, h_2^2\right\} \right) \times \left[\frac{4h_1h_2}{h_1^2 + h_2^2} + C_4 - C_5(h_1 + h_2) + 2\pi \ln \left| \frac{\sqrt{2}((\max\{L_1, L_2\} - \min\{h_1, h_2\}))}{\min\{h_1, h_2\}} \right| \right]^{\frac{1}{2}};$$

- *for* $2 < p < \infty$*:*

$$\left\| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^p} \leq \left(C_1 \max\{h_1, h_2\} + C_2 \frac{\max\left\{h_1^2, h_2^2\right\}}{\min\{h_1, h_2\}} + C_3 \max\left\{h_1^2, h_2^2\right\} \right) \times \left[\frac{4h_1h_2}{(h_1^2 + h_2^2)^{\frac{p}{2}}} + \frac{2h_2}{p-1}\left(2h_1^{1-p} + ph_1^{1-p} - \frac{3}{L_1^{p-1}}\right) + \frac{2h_1}{p-1}\left(2h_2^{1-p} + ph_2^{1-p} - \frac{3}{L_2^{p-1}}\right) + \frac{2\pi}{p-2}\left((\min\{h_1, h_2\})^{p-2} - \left(\sqrt{2}(\max\{L_1, L_2\} - \min\{h_1, h_2\})\right)^{p-2} \right) \right]^{\frac{1}{p}};$$

- *for* $p = \infty$*:*

$$\| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \|_{l^\infty} \leq \frac{1}{\sqrt{\min\left\{h_1^2, h_2^2\right\}}} \left(C_1 \max\{h_1, h_2\} + C_2 \frac{\max\left\{h_1^2, h_2^2\right\}}{\min\{h_1, h_2\}} + C_3 \max\left\{h_1^2, h_2^2\right\} \right);$$

where all constants are independent on the stepsizes h_1 *and* h_2*.*

Proof. The proof of the theorem is analogous to the proofs of Theorem 2. □

Similar, to the l^p-estimates for the discrete fundamental solution $E^{(1)}_{h_1,h_2}$, Figure 6 illustrates the estimates presented in Theorem 4. The estimates are calculated for different values of p, a rectangular domain with side lengths $L_1 = 1$, $L_2 = 2$ is discretised by a lattice with $h_2 = \alpha h_1$ for $\alpha = 3$. As it can be clearly seen, all estimates converge to zero for $h_1 \to 0$, as expected, and the l^∞-estimate is bounded, as one could expect as well.

Short forms of the l^p-estimates from Theorem 4 are provided in the following corollary:

Corollary 5. *Under assumptions of Theorem 4 and assuming* $h_2 = \alpha h_1$ *for* $\alpha \in \mathbb{R}$*, the following estimates hold:*

- *for* $p = 1$*:* $\left\| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^1} \leq C_1(\alpha)h_1 + C_2(\alpha)h_1^2 + C_3h_1^3;$

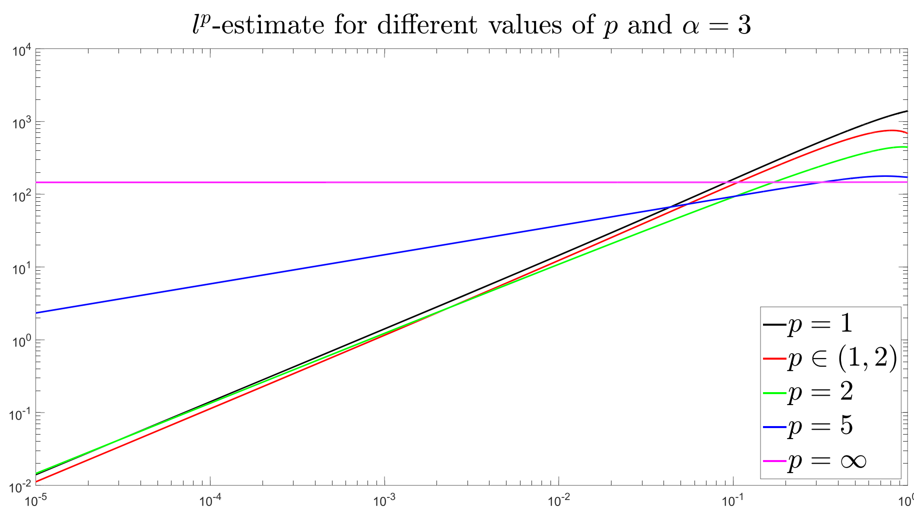

FIGURE 6 Calculation of the $l^p(\Omega_{h_1,h_2})$-error estimate from Theorem 4 in a logarithmic scale with respect to h_1 for $h_2 = 3h_1$ for a rectangular domain with length $L_1 = 1$ and height $L_2 = 2$ [Colour figure can be viewed at wileyonlinelibrary.com]

- *for* $1<p<2$: $\left\|E^{(2)}_{h_1,h_2}(\mathbf{x})-E(\mathbf{x})\right\|_{l^p} \leq \left(C_1(\alpha)h_1+C_2(\alpha)h_1^2\right)\left(C_3(\alpha,p)h_1^{2-p}-C_4(\alpha,p)h_1\right)^{\frac{1}{p}}$;
- *for* $p=2$: $\left\|E^{(2)}_{h_1,h_2}(\mathbf{x})-E(\mathbf{x})\right\|_{l^2} \leq \left(C_1(\alpha)h_1+C_2(\alpha)h_1^2\right)\left(C_3(\alpha)-C_4(\alpha)h_1-2\pi\ln h_1\right)^{\frac{1}{2}}$;
- *for* $2<p<\infty$: $\left\|E^{(2)}_{h_1,h_2}(\mathbf{x})-E(\mathbf{x})\right\|_{l^p} \leq \left(C_1(\alpha)h_1+C_2(\alpha)h_1^2\right)\left(C_3(\alpha,p)h_1^{p-2}-C_4(\alpha,p)h_1\right)^{\frac{1}{p}}$;
- *for* $p=\infty$: $\|E^{(2)}_{h_1,h_2}(\mathbf{x})-E(\mathbf{x})\|_{l^\infty} \leq C_1(\alpha)+C_2(\alpha)h_1$,

where some of the constants depend on α *and p, while the other depend only on* α*, and one constant for* $p=1$ *does not depend on* α *and p. Moreover, all constant depending on* α *tend to infinity for* $\alpha \to 0$ *and* $\alpha \to \infty$.

Next step is to construct l^p-estimates in the exterior domain, which is now possible, as it has been mentioned above, since the discrete fundamental solution $E^{(2)}_{h_1,h_2}$ is considered. The following theorem presents the l^p-estimates for the exterior domain:

Theorem 5. *Let* $E^{(2)}_{h_1,h_2}$ *be the discrete fundamental solution given in (13) of the discrete Laplace operator, and let E be the continuous fundamental solution (3) of the classical Laplace operator. Let* Ω_{h_1,h_2} *be a discrete domain symmetric with respect to coordinate origin, and let* $\Omega^{ext}_{h_1,h_2}$ *be its exterior domain. Let* L_1 *be the maximal distance between the coordinate origin and boundary of* Ω_{h_1,h_2} *in* x_1 *direction, and* L_2 *respectively be the maximal distance between the coordinate origin and boundary of* Ω_{h_1,h_2} *in* x_2 *direction. Then for all* $h_1,h_2<\sqrt{2}\pi$ *the following estimate in* $l^p(\Omega^{ext}_{h_1,h_2})$ *holds for* $p>2$:

$$\left\|E^{(2)}_{h_1,h_2}-E\right\|_{l^p} \leq \left(C_1\max\{h_1,h_2\}+C_2\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}}+C_3\max\left\{h_1^2,h_2^2\right\}\right)\times\left(\frac{4h_1h_2}{\left(L_1^2+L_2^2\right)^{\frac{p}{2}}}\right.$$
$$\left.+\frac{4h_2}{L_1^{p-1}(p-1)}+\frac{4h_1}{L_2^{p-1}(p-1)}+\frac{2\pi}{p-2}(\min\{L_1,L_2\})^{2-p}+\frac{2h_1(2L_2-h_2)}{(L_1+h_1)^p}\cdot\frac{p+l_1}{p-1}+\frac{2h_2(2L_1-h_1)}{(L_2+h_2)^p}\cdot\frac{p+l_2}{p-1}\right)^{\frac{1}{p}},$$

and for $p=\infty$:

$$\left\|E^{(2)}_{h_1,h_2}(\mathbf{x})-E(\mathbf{x})\right\|_{l^\infty} \leq \frac{1}{\min\{L_1+h_1,L_2+h_2\}}\left(C_1\max\{h_1,h_2\}+C_2\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}}+C_3\max\left\{h_1^2,h_2^2\right\}\right),$$

where all constants are independent on the stepsizes h_1 *and* h_2.

Proof. After using the Minkowski inequality, the following expression is obtained for a short form of the estimate:

$$\left\|E^{(2)}_{h_1,h_2}-E\right\|_{l^p} = \left(\sum_{(m_1h_1,m_2h_2)\in\Omega^{ext}_{h_1,h_2}}\left|E^{(2)}_{h_1,h_2}(m_1h_1,m_2h_2)-E(m_1h_1,m_2h_2)\right|^p h_1h_2\right)^{\frac{1}{p}}$$
$$\leq \left\|\frac{1}{|\mathbf{x}|}\right\|_{l^p}\left(C_1\max\{h_1,h_2\}+C_2\frac{\max\left\{h_1^2,h_2^2\right\}}{\min\{h_1,h_2\}}+C_3\max\left\{h_1^2,h_2^2\right\}\right).$$

Assumption of the theorem, that the discrete domain Ω_{h_1,h_2} is a rectangular domain symmetric with respect to the coordinate origin is necessary for carrying out the proof explicitly. The use of this theorem for domains of a general shape will be discussed later. Applying the definition of the l^p-norm to the term $\frac{1}{|\mathbf{x}|}$, and taking into account that exterior domain is considered, the following expression is obtained

$$\left\|\frac{1}{|\mathbf{x}|}\right\|_{l^p} = \left(4\sum_{m_1=l_1}^{\infty}\sum_{m_2=l_2}^{\infty}\frac{h_1h_2}{(m_1^2h_1^2+m_2^2h_2^2)^{\frac{p}{2}}}+4\sum_{m_1=l_1+1}^{\infty}\sum_{m_2=1}^{l_2-1}\frac{h_1h_2}{(m_1^2h_1^2+m_2^2h_2^2)^{\frac{p}{2}}}\right.$$
$$\left.+4\sum_{m_1=1}^{l_1-1}\sum_{m_2=l_2+1}^{\infty}\frac{h_1h_2}{(m_1^2h_1^2+m_2^2h_2^2)^{\frac{p}{2}}}+2\sum_{m_1=l_1+1}^{\infty}\frac{h_1h_2}{(m_1h_1)^p}+2\sum_{m_2=l_2+1}^{\infty}\frac{h_1h_2}{(m_2h_2)^p}\right)^{\frac{1}{p}}. \tag{14}$$

Figure 7 shows a subdivision of the exterior domain, which simplifies constructions of the estimate. Thus, the first series corresponds to the region I, where the fact that exterior corner points belong to the exterior domain has been taken into account.[9] The second and the third series correspond to the strips II and III, respectively. The last two series represent summations along coordinate axes. It is also necessary to mention, that dimensions of the interior domain Ω_{h_1,h_2} are fixed, i.e., coordinates of the points (L_1, L_2), $(-L_1, L_2)$, $(L_1, -L_2)$, $(-L_1, -L_2)$, because exterior domain $\Omega^{ext}_{h_1,h_2}$ is considered. However, indices of points corresponding to the discrete boundary layer $\gamma^-_{h_1,h_2} = \alpha^-_{h_1,h_2}$ (edges of the rectangle in Figure 7) depend on the stepsizes as $l_1 = \frac{L_1}{h_1}$ and $l_2 = \frac{L_2}{h_2}$. This dependency will be addressed at the last step of the proof, while indices of the corresponding points will be used during the proof.

Let us estimate the terms in (14) by help of the integral test. The estimates will be done for $p > 2$, because the error for $p \in [1, 2]$ does not converge to zero in the case of unbounded domains even for h_1 and h_2 tending to zero. The single series in (14) can be estimated as follows:

$$\sum_{m_1=l_1+1}^{\infty} \frac{h_1h_2}{(m_1h_1)^p} \leq \frac{h_1h_2}{(l_1+1)^p h_1^p} + \int_{l_1+1}^{\infty} \frac{h_1h_2}{(m_1h_1)^p} dm_1 \leq \frac{h_1h_2}{(l_1+1)^p h_1^p}\left(1 + \frac{l_1+1}{p-1}\right),$$
$$\sum_{m_2=l_2+1}^{\infty} \frac{h_1h_2}{(m_2h_2)^p} \leq \frac{h_1h_2}{(l_2+1)^p h_2^p} + \int_{l_2+1}^{\infty} \frac{h_1h_2}{(m_2h_2)^p} dm_2 \leq \frac{h_1h_2}{(l_2+1)^p h_2^p}\left(1 + \frac{l_2+1}{p-1}\right).$$

To construct the estimate for the double series corresponding to the region II, the fact that this region is, in fact, can be represented as the product

$$II := [l_1 + 1, \infty) \times [1, l_2 - 1]$$

will be used. Thus, the estimate along the x_1-axis needs to be multiplied with amount such lines appearing in the region II, which is equal to $l_2 - 1$. Thus, the following estimate for the region II is obtained:

$$\begin{aligned}\sum_{m_1=l_1+1}^{\infty}\sum_{m_2=1}^{l_2-1} \frac{h_1h_2}{(m_1^2h_1^2 + m_2^2h_2^2)^{\frac{p}{2}}} &\leq \sum_{m_1=l_1+1}^{\infty} \frac{h_1h_2}{(m_1h_1)^p}\cdot(l_2-1) \leq \frac{h_1h_2}{(l_1+1)^p h_1^p}\left(1 + \frac{l_1+1}{p-1}\right)\cdot\left(\frac{L_2}{h_2} - 1\right) \\ &\leq \frac{L_2h_1}{(L_1+h_1)^p}\left(1 + \frac{l_1+1}{p-1}\right) - \frac{h_1h_2}{(L_1+h_1)^p}\left(1 + \frac{l_1+1}{p-1}\right),\end{aligned}$$

where the facts that $L_1 = h_1l_1$ and $L_2 = h_2l_2$ have been used. Similarly, the estimate for the region III can be obtained:

$$\begin{aligned}\sum_{m_1=1}^{l_1-1}\sum_{m_2=l_2+1}^{\infty} \frac{h_1h_2}{(m_1^2h_1^2 + m_2^2h_2^2)^{\frac{p}{2}}} &\leq \sum_{m_2=l_2+1}^{\infty} \frac{h_1h_2}{(m_2h_2)^p}\cdot(l_1-1) \leq \frac{h_1h_2}{(l_2+1)^p h_2^p}\left(1 + \frac{l_2+1}{p-1}\right)\cdot\left(\frac{L_1}{h_1} - 1\right) \\ &\leq \frac{L_1h_2}{(L_2+h_2)^p}\left(1 + \frac{l_2+1}{p-1}\right) - \frac{h_1h_2}{(L_2+h_2)^p}\left(1 + \frac{l_2+1}{p-1}\right).\end{aligned}$$

Next step is to estimate the series related to the region I in Figure 7. Application of the integral test to the double series gives:

$$\sum_{m_1=l_1}^{\infty}\sum_{m_2=l_2}^{\infty} \frac{h_1h_2}{(m_1^2h_1^2 + m_2^2h_2^2)^{\frac{p}{2}}} \leq \frac{h_1h_2}{(l_1^2h_1^2 + l_2^2h_2^2)^{\frac{p}{2}}} + \int_{l_1h_1}^{\infty} \frac{h_2}{(x^2 + l_2^2h_2^2)^{\frac{p}{2}}} dx + \int_{l_2h_2}^{\infty} \frac{h_1}{(l_1^2h_1^2 + y^2)^{\frac{p}{2}}} dy + \int_{l_1h_1}^{\infty}\int_{l_2h_2}^{\infty} \frac{1}{(x^2+y^2)^{\frac{p}{2}}} dydx.$$

To estimate the double integral in the above expression, the transformation to polar coordinates by extending the rectangular lattice to the biggest possible square lattice is used. Thus, the following estimate is obtained:

$$\int_{l_1h_1}^{\infty}\int_{l_2h_2}^{\infty} \frac{1}{(x^2+y^2)^{\frac{p}{2}}} dydx \leq \int_{\min\{l_1h_1, l_2h_2\}}^{\infty}\int_{\min\{l_1h_1, l_2h_2\}}^{\infty} \frac{1}{(x^2+y^2)^{\frac{p}{2}}} dydx \leq \int_{\min\{l_1h_1, l_2h_2\}}^{\infty}\int_{0}^{\frac{\pi}{2}} \frac{1}{r^p} r\,d\varphi\,dr = \frac{\pi(\min\{l_1h_1, l_2h_2\})^{2-p}}{2(p-2)}.$$

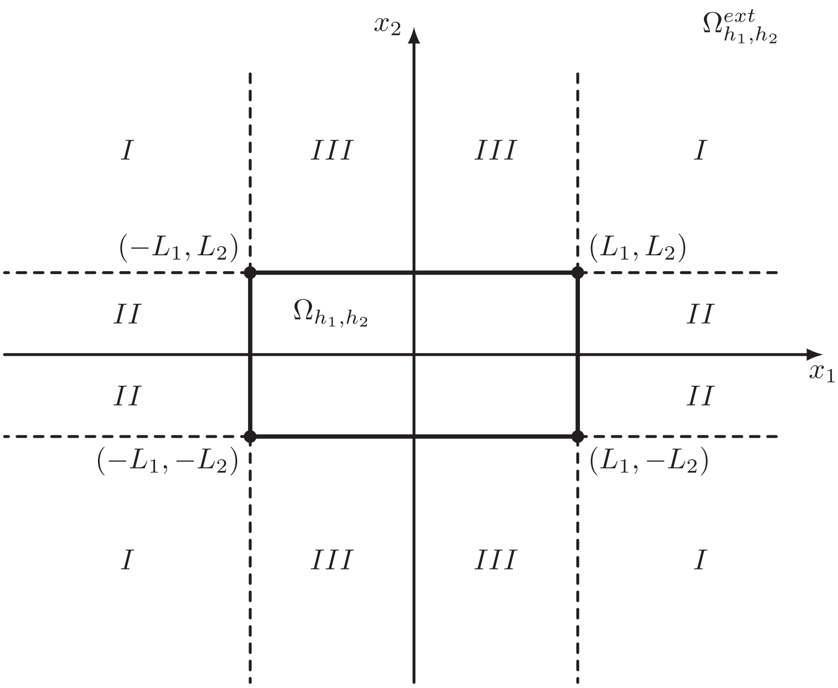

FIGURE 7 Subdivision of the exterior domain $\Omega^{ext}_{h_1,h_2}$ for constructing l^p-estimates

The one-dimensional integrals can be estimated as follows:

$$\int_{l_1h_1}^{\infty} \frac{h_2}{(x^2+l_2^2h_2^2)^{\frac{p}{2}}}dx \leq \frac{h_2}{(p-1)(l_1h_1)^{p-1}}, \qquad \int_{l_2h_2}^{\infty} \frac{h_1}{(l_1^2h_1^2+y^2)^{\frac{p}{2}}}dy \leq \frac{h_1}{(p-1)(l_2h_2)^{p-1}}.$$

Combining all considerations presented above, the following estimate is finally obtained:

$$\left\|\frac{1}{|\mathbf{x}|}\right\|_{l^p} \leq \left(\frac{4h_1h_2}{\left(L_1^2+L_2^2\right)^{\frac{p}{2}}} + \frac{4h_2}{L_1^{p-1}(p-1)} + \frac{4h_1}{L_2^{p-1}(p-1)} + \frac{2\pi}{p-2}(\min\{L_1,L_2\})^{2-p} \right.$$
$$\left. + \frac{2h_1(2L_2-h_2)}{(L_1+h_1)^p}\cdot\frac{p+l_1}{p-1} + \frac{2h_2(2L_1-h_1)}{(L_2+h_2)^p}\cdot\frac{p+l_2}{p-1}\right)^{\frac{1}{p}}.$$

The proof of l^∞-estimate is analogous to the proof of Theorem 2, and it needs to be taken into account that $\frac{1}{|\mathbf{x}|}$ has its maximum on the interior boundary layer $\alpha^+_{h_1,h_2}$ of the exterior domain, which corresponds to minimum indices of the point of exterior domain. This boundary layer can be characterised by using distances L_1 and L_2, as it has been done in Theorem 4, and making shifts towards exterior in corresponding directions. Thus, the points of $\alpha^+_{h_1,h_2}$ are characterised by the distances $|L_1+h_1|$ and $|L_2+h_2|$ in the x_1 and x_2 directions, respectively. The rest of the proof follows immediately. Thus, the theorem is proved. □

Figure 8 illustrates the estimates presented in Theorem 5 for the exterior of a rectangular domain with side lengths $L_1 = 1$, $L_2 = 2$, which is discretised by a lattice with $h_2 = \alpha h_1$ for $\alpha = 3$. As it can be clearly seen, both estimates converge to zero for $h_1 \to 0$, as expected, and thus indicating the advantage of working with the reformulated discrete fundamental solution $E^{(2)}_{h_1,h_2}$.

Remark 1. It is necessary to remark how the l^p-estimates for interior and exterior domains presented in this section can be used for discrete domains of arbitrary shape. Consider for example an L-shape domain, which is not symmetric with respect to the coordinate origin. In order to apply the l^p-estimates presented in this section, the L-shape domain should be replaced by the smallest possible rectangular domain containing the original L-shape domain, and the coordinate origin should be placed at its centre of symmetry. After that, all estimates can be used directly. Of course in this case the estimates will be rough estimates, and they become worse for domains elongated in one direction.

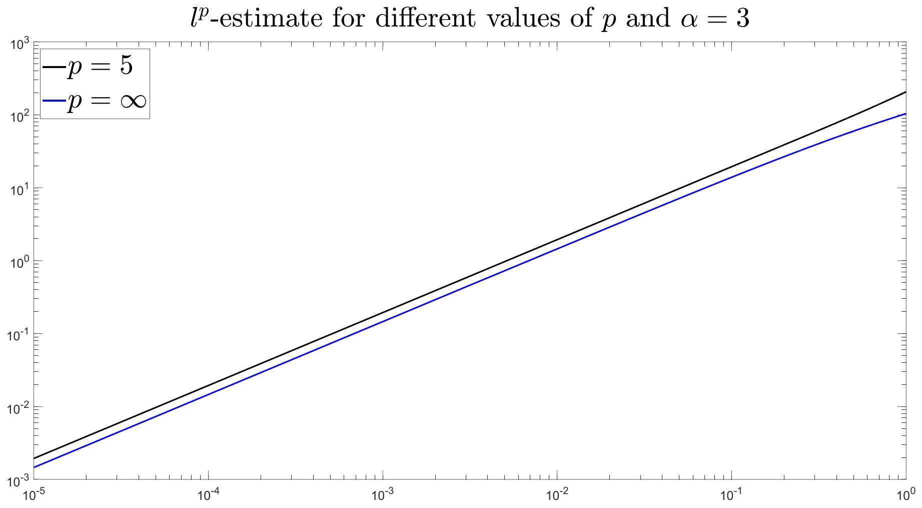

FIGURE 8 Calculation of the $l^p(\Omega^{ext}_{h_1,h_2})$-error estimates from Theorem 5 in a logarithmic scale with respect to h_1 for $h_2 = 3h_1$ for the exterior of a rectangular domain with length $L_1 = 1$ and height $L_2 = 2$ [Colour figure can be viewed at wileyonlinelibrary.com]

Nonetheless, this approach provides first ideas for error analysis of arbitrary-shaped discrete domains, since explicit calculations of error estimates, as presented in this section, can be carried out only for some specific case, and not in the general case.

The following corollary presents short forms of the l^p-estimates in the exterior domain:

Corollary 6. *Under assumptions of Theorem 5 and assuming $h_2 = \alpha h_1$ for $\alpha \in \mathbb{R}$, the following estimates hold:*

- *for $2 < p < \infty$: $\left\| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \right\|_{l^p} \leq \left(C_1(\alpha)h_1 + C_2(\alpha)h_1^2 \right) \left(C_3(\alpha,p)h_1^2 + C_4(\alpha,p)h_1 + C_5(p) \right)^{\frac{1}{p}}$;*
- *for $p = \infty$: $\| E^{(2)}_{h_1,h_2}(\mathbf{x}) - E(\mathbf{x}) \|_{l^\infty} \leq C_1(\alpha)h_1 + C_2(\alpha)h_1^2$,*

where some of the constants depend on α and p, while the other depend only on α or p. Moreover, all constant depending on α tend to infinity for α → 0 and α → ∞.

4 | CONCLUSIONS

Several estimates for the discrete fundamental solution of the discrete Laplace operator on a rectangular lattice have been presented in this paper. Additionally to extension of the classical estimates, new results related to the exterior setting have been constructed. Thus, this article provides a basis for future convergence analysis of the discrete potential method on a rectangular lattice.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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REFERENCES

1. Hommel A. *Fundamentallösungen partieller Differenzenoperatoren und die Lösung diskreter Randwertprobleme mit Hilfe von Differenzenpotentialen*. PhD thesis, Bauhaus-Universität Weimar, Weimar, Germany, 1998.
2. Ryaben'kii VS. *Method of Difference Potentials and its Applications*. Berlin Heidelberg: Springer-Verlag; 2002.

3. Beale JT, Ying W. Solution of the Dirichlet problem by a finite difference analog of the boundary integral equation. *Numerische Mathematik*. 2019;141(3):605-626.
4. Faustino N, Gürlebeck K, Hommel A, Kähler U. Difference potentials for the Navier–Stokes equations in unbounded domains. *J Differ Equ Appl*. 2006;12(6):577-595.
5. Gürlebeck K, Hommel A. On finite difference potentials and their applications in a discrete function theory. *Math Methods Appl Sci*. 2002;25(16-18):1563-1576.
6. Ryaben'kii VS. *Difference Potentials Method for Some Problems of Continuous Media Mechanics*. Moscow: Nauka; 1987 (in Russian).
7. Tsynkov SV. On the definition of surface potentials for finite-difference operators. *J Sci Comput*. 2003;18(2):155-189.
8. Gürlebeck K, Hommel A, Legatiuk A. Methods of discrete potential theory for an induction heating problem, *AIP Conference Proceeding* 1978, 280003, 2018.
9. Gürlebeck K, Hommel A, Legatiuk A. Geometry discretisation algorithm for finite difference operators arising in discrete potential and function theories, *AIP Conference Proceeding*, in press, 2020.
10. Legatiuk A, Gürlebeck K, Hommel A. The discrete Borel-Pompeiu formula on a rectangular lattice. *Adv Appl Clifford Algebra*. 2018;28(3):69.
11. Thomée V. Discrete interior Schauder estimates for elliptic difference operators. *SIAM J Numer Anal*. 1968;5(3):626-645.
12. Vladimirov VS. *Methods of the Theory of Generalized Functions*. Taylor & Francis; 2002.
13. Stummel F. Elliptische Differenzenoperatoren unter Dirichletrandbedingungen. *Mathematische Zeitschrift*. 1967;97(3):169-211.
14. Schumaker L. *Spline Functions on Triangulation*. Cambridge University Press; 2007.
15. Gürlebeck K, Sprößig W. *Quaternionic and Clifford calculus for engineers and physicists*. Chichester: John Wiley & Sons; 1997.
16. Gradshteyn IS, Ryzhik IM. *Table of Integrals, Series, and Products*. Seventhed. Elsevier; 2007.

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